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LETTER FROM THE EDITOR

Dear Readers,

In 2020, we started to publish papers only in English. The current volume is the 13th in the series. So far, we have published 38 research papers, 5 occasional papers and 4 conference reports. They have been viewed over 23,000 times on our web page. We would like to thank you for your growing interest in *Przegląd Statystyczny. Statistical Review*. To make the journal more visible, this year we expanded its abstracting and indexing by the Directory of Open Access Journals (DOAJ).

Meanwhile, we are seeking new submissions. We welcome high-quality papers addressing significant issues from various branches of economics, finance and management, by all interested authors, including PhD candidates. Articles on theoretical and empirical topics in statistics, econometrics, mathematical economics, operational research, decision sciences and data analysis are particularly welcome. The full editorial process – from the paper's submission to its publishing – is free of charge. The final decision regarding the publication of a paper is issued within approximately two months.

On behalf of the Board of Editors,
Paweł Miłobędzki
Editor-in-Chief

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Ascending Probabilistic Max-min Extended Choice Correspondence

Somdeb Lahiri[a]

Abstract. In this paper, we provide an axiomatic characterization of the ascending probabilistic max-min extended choice correspondence for a decision-maker who has state-dependent preferences (represented by a linear order) over a set of alternatives and a (subjective) probability vector over states of nature, where both the preferences and probability vectors are variable. Further on the domain of all extended preference profiles for which the Ascending Probabilistic Max-Min Extended Choice Correspondence is resolute, the same choice correspondence is completely characterized by just two of the three axioms that are required for the axiomatic characterization on the more general domain. A significant feature of our solution concept and the related axiomatic analysis is that we use no more information than the probability with which each alternative realizes each rank.

Keywords: decision-making under risk, state-dependent preferences, extended choice correspondences, ascending probabilistic max-min
JEL: C02, D81, D91

1. Introduction

The framework of decision-making under uncertainty, introduced in Lahiri (2020/2021, 2022), is that of a decision-maker who is faced with making a choice under probabilistic uncertainty (risk) regarding the future state of nature, which is realized after the decision has been made. The decision-maker is provided with (or aware of) an extended preference profile, which is a pair whose first component is a profile of state-dependent rankings over a non-empty finite set of alternatives (the consequences) and whose second component is a probability distribution over a non-empty finite set of states of nature. A decision support system (DSS) or decision aid is required to choose a non-empty "desirable" set of alternatives from which the final choice has to be made. The decision aid or DSS has no bias in favor of any one or more alternatives that it suggests. Such a decision support system is called an extended choice correspondence, i.e. a rule which associates each extended preference profile from a given set of extended preference profiles with a non-empty finite set of desirable alternatives. For related literature, one might wish to consult Lahiri (2022).

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Here we begin by setting up the model for extended choice correspondences. In this framework, we define and provide an axiomatic characterization of the Ascending Probabilistic Max-min Extended Choice Correspondence, which is a refined version of the Probabilistic Max-min Choice Correspondence defined and axiomatically characterized in Lahiri (2022). The Probabilistic Max-min Extended Choice Correspondence is based on the Max-min Choice Correspondence defined in Campbell, Kelly and Qi (2018). This choice correspondence selects, for each preference profile, those alternatives which have the best "worst rank". In our framework, for an extended preference profile – a pair comprising of a strict preference profile and a probability vector (for the states of nature) – a "max-min alternative" is an alternative whose worst rank among states of nature that occur with positive probability is the best. The worst rank of a max-min alternative is said to be the "max-min rank". The probabilistic max-min extended choice correspondence selects, for each extended preference profile, those max-min alternatives which have the least positive probability of attaining the "max-min rank". We ignore those states of nature which occur with zero probability, since if an alternative attains its worst rank with zero probability, it is improbable (though not impossible) that it will attain such a rank. Further, if a max-min alternative attains the max-min rank with the lowest probability, then it attains a superior rank with the highest probability among all the max-min alternatives. The Ascending Probabilistic Max-min Extended Choice Correspondence chooses those alternatives from among the Probabilistic Max-min winners that occur with the greatest (cumulative) probability of a better rank, as we keep improving the rank, one rank each time, and stop as soon as we arrive at a unique solution, or the moment we reach the first rank – whichever happens sooner. It is very unlikely that a risk-averse individual to whom the probabilistic max-min extended choice correspondence is recommended would wish for anything better. Hence, the solution studied here unconditionally supersedes the solution presented in Lahiri (2022).

The axioms we use to characterize the Ascending Probabilistic Max-min Extended Choice Correspondence are Independence of Irrelevant States, Probabilistic Neutrality and No-Terminal Stochastic Domination. Further on the domain of all extended preference profiles for which the Ascending Probabilistic Max-Min Extended Choice Correspondence is resolute, the same choice correspondence is completely characterized by Independence of Irrelevant States and No-Terminal Stochastic Domination. Probabilistic Neutrality is no longer required for the axiomatic characterization on such a domain.

The Independence of Irrelevant States says that states of nature that occur with zero probability have no influence or effect on the choice procedure. Probabilistic Neutrality says that if two alternatives have identical probabilities of realizing each and every rank, then either both are chosen or neither is chosen. An alternative is said to terminally stochastically dominate another alternative if there is a rank such that the probability of the first alternative getting that rank or better is higher than the probability of the second alternative getting the same rank or better, and for all worse ranks, the probability of the first alternative getting that rank or better is no less than the probability of the second alternative getting the same rank or better. In other words, towards the end the first alternative has a better chance of having a preferred position than the second alternative. No-Terminal Stochastic Domination says that a terminally stochastically dominated alternative is not chosen. The interesting characteristic of our result is that we are able to obtain it without any axiom appealing to worst ranks, although the worst rank is one of the most important features – in fact the starting point – in the definition of our solution concept. A significant feature of our solution concept and the related axiomatic analysis is that we use no more information than the probability with which each alternative realizes each rank.

The domain of the ascending probabilistic max-min extended choice correspondence whose axiomatization we provide is the set of all extended preference profiles such that, for any non-empty subset of probability vectors, all strict preference profiles can be associated with any probability vector in the subset. However, the axiomatic characterization we provide continues to remain valid on the strict sub-domain where the extended preference profiles are such that those states of nature that occur with positive probability have an equal probability of occurrence. Such a domain is called a domain with equiprobable support. As regards the domain with equiprobable support, our solution concept is a strict refinement of the one discussed in Campbell, Kelly and Qi (2018), with a different interpretation.

2. The framework of the analysis

The following framework, which is identical to the one in Lahiri (2022), is a relatively close adaptation of the ones from Denicolò (1985), Section 2.2 of Endriss (2011) and those discussed thoroughly in Lahiri (2020/2021). There are passages in Sections 2 and 3 of the latter paper where the wording is identical to some passages in Lahiri (2022). It is necessary to include them, since unlike the results which can be referred to, these passages are concerned with basic notations and definitions, and it would be a harassment for the readers to ask them to look for those definitions elsewhere. However, all such passages have been included between inverted commas in what follows.

"Consider a decision-maker (DM) faced with the problem of choosing one or more alternatives from a non-empty finite set of alternatives X, containing at least three elements. Let $\Psi(X)$ denote the set of all non-empty subsets of X. For a positive integer $n \geq 3$, let $N = \{1,2,\dots,n\}$. In contrast to the convention, we will refer to an element in N as a state of nature and the set N as the set of states of nature.

A strict preference relation/strict ranking on X is a linear order (i.e. a reflexive, complete/connected/total, transitive and anti-symmetric binary relation) on X. Generally, a strict preference relation is denoted by R with P signifying its asymmetric part. If for $x, y \in$ X it is the case that $(x, y) \in$ R, then we shall denote it by xRy and say that x is at least as good as y for the strict preference relation R. Similarly, xPy interpreted as x is strictly preferred to y for the strict preference relation R.

Given a strict preference R and an alternative x, the rank of x at R denoted $\text{rk}(x, \text{R}) = |\{y \in \text{X} | y\text{R}x\}|$, i.e. 1 + cardinality of the set of alternatives strictly preferred to x for the strict preference relation R.

Let $\mathcal{L}$ denote the set of all strict preference relations on X."

A convenient way to display/represent a strict ranking R is by using an m-dimensional column vector $\begin{pmatrix} x \\ \vdots \\ z \end{pmatrix}$, such that the entry in the r^th^ row corresponds to the alternative that has the r^th^ rank at the strict ranking R.

"A strict preference profile denoted R_N is a function from N to $\mathcal{L}$. R_N is represented as the array $\langle \text{R}_\text{i} | \text{i} \in \text{N} \rangle$, where R_i is the strict preference relation/strict ranking in state of nature i. The set of all preference profiles is denoted $\mathcal{L}^\text{N}$.

A probability vector over N is a vector $\text{p} \in \mathbb{R}_+^\text{N}$ satisfying $\sum_{\text{i}=1}^{\text{N}} \text{p}_\text{i} = 1$, where for i$\in$N, p_i is the probability that state of nature i occurs.

The set of probability vectors over N is denoted by Δ.

Given a probability vector p, the set $\{\text{j} | \text{p}_\text{j} > 0\}$ is referred to as the support of p and denoted support(p).

Since probabilities are associated with events, for each i$\in$N, the state of nature i represents a non-empty set, and N is a finite partition of some underlying sample space.

A pair $(\text{R}_\text{N},\text{p}) \in \mathcal{L}^\text{N} \times \Delta$ is said to be an extended preference profile and $\mathcal{L}^\text{N} \times \Delta$ is the set of all extended preference profiles.

Given $(\text{R}_\text{N},\text{p}) \in \mathcal{L}^\text{N} \times \Delta$ and an alternative x (i.e. x$\in$X), the state of nature i (i.e. i$\in$N) is referred to as the worst state of nature for x at $(\text{R}_\text{N},\text{p})$ if $\text{i} \in \underset{\text{j} \in \text{support(p)}}{\text{argmax}} \ \text{rk}(\text{x}, \text{R}_\text{j})$.

The definition above says that a state of nature is the worst state of nature for an alternative if the state of nature occurs with "positive probability", and the alternative does not attain any worse rank with "positive probability".

Given $(\text{R}_\text{N},\text{p}) \in \mathcal{L}^\text{N} \times \Delta$ and an alternative x (i.e. x$\in$X), the set $\text{WS}(\text{x}, (\text{R}_\text{N},\text{p})) = \{\text{i} |$ i is the worst state of nature for x$\}$ is said to be the set of the worst states of nature for x at $(\text{R}_\text{N},\text{p})$, and for $\text{i} \in \text{WS}(\text{x}, (\text{R}_\text{N},\text{p}))$, $\text{rk}(\text{x},\text{R}_\text{i})$ denoted $\text{worstrk}(\text{x},(\text{R}_\text{N},\text{p}))$ is said to be the worst rank of x at $(\text{R}_\text{N},\text{p})$.

Clearly, $\text{worstrk}(\text{x},(\text{R}_\text{N},\text{p})) = \max\{\text{rk}(\text{x},\text{R}_\text{i}) | \text{i} \in \text{support(p)}\}$ for all x$\in$X.

For all $(R_N,p)\in\mathcal{L}^N\times\Delta$, let $Mm(R_N,p) = \underset{y\in X}{\text{argmin}}\, \text{worstrk}(y,(R_N,p))$.

$Mm(R_N,p)$ is said to be the set of max-min alternatives at (R_N,p). The max-min rank for (R_N,p) is equal to the unique worstrk$(x,(R_N,p))$ for any $x\in Mm(R_N,p)$.

A domain is any non-empty subset of $\mathcal{L}^N\times\Delta$. We will denote a domain by $\mathcal{R}$.

An extended choice correspondence (ECC) on (domain) $\mathcal{R}$ is a function f from $\mathcal{R}$ to $\Psi(X)$."

Useful Notations: Given $(R_N,p)\in \mathcal{L}^N\times\Delta$, $x\in X$ and $r\in\{1,\ldots,m\}$:

(a) Let $Pr(\{rk(x) = r\}|(R_N,p))$ denote the probability of x being ranked r^{th} at (R_N,p), which is equal to $\sum_{\{j\in N|rk(x,R_j)=r\}} p_j$.

(b) Let $Pr(\{rk(x) \leq r\}|(R_N,p))$ denote the probability of x being ranked "r^{th} or better" at (R_N,p), which is equal to $\sum_{\{j\in N|rk(x,R_j)\leq r\}} p_j$.

Given that $(R_N,p)\in \mathcal{L}^N\times\Delta$, $x\in X$ and $r\in\{2,\ldots,m\}$, let $Pr(\{rk(x) < r\}|(R_N,p))$ denote the probability of x being ranked "better than r^{th}" at (R_N,p), which is equal to $\sum_{\{j\in N|rk(x,R_j)< r\}} p_j$.

An ECC on (domain) $\mathcal{R}$ is said to be resolute if it is singleton valued for all preference profiles on $\mathcal{R}$.

3. Some axioms and a lemma which will be useful on the way

"In what follows, we will be concerned only with those domains which satisfy the following property:

Domain Property: $R=\mathcal{L}^N\times Q$, where Q is a non-empty subset of Δ."

The following is a desirable axiom that few would wish to contest.

An ECC f on $\mathcal{R}$ is said to satisfy Independence of Irrelevant States (be Independent of Irrelevant of States) (IIS) if for all (R_N,p), $(R'_N,p)\in \mathcal{R}$: $[\{j|p_j > 0\} \subset \{j|R_j = R'_j\}]$ implies $[f(R'_N,p) = f(R_N,p)]$.

In view of (IIS) and the issues we will be concerned with here – which depend only on the probability with which each strict ranking occurs – an alternative way of displaying an extended preference profile is equally convenient.

If for $(R_N,p)\in\mathcal{R}$ there exists a positive integer K such that a strict ranking $R = R_j$ for $j\in$ support (p) if and only if $R\in\{R_{(1)},\ldots, R_{(K)}\}$, then (R_N,p) can be displayed as:

$\begin{array}{ccccc} p_{(1)} & \cdots & p_{(k)} & \cdots & p_{(K)} \end{array}$
$[R_{(1)} \;\cdots\; R_{(k)} \;\cdots\; R_{(K)}]$, where $[R_{(1)} \;\cdots\; R_{(k)} \;\cdots\; R_{(K)}]$ is an $m \times K$ matrix such that for $k\in\{1,\ldots,K\}$, the k^{th} column is the column vector representing the strict ranking $R_{(k)}$ and the $p_{(k)}$ on top of the k^{th} column denotes the probability with which the state of nature is such that the strict ranking $R_{(k)}$ is realized, i.e., $p_{(k)} =\sum_{\{j\in support(p)|R_j= R_{(k)}\}} p_j$.

An ECC f on $\mathcal{R}$ is said to satisfy Probabilistic Neutrality if for all $(R_N,p)\in \mathcal{R}$ and $x,y\in X$: $[\sum_{\{j\in N|rk(y,R_j)=r\}} p_j = \sum_{\{j\in N|rk(x,R_j)=r\}} p_j$ for all $r\in\{1,\ldots,m\}]$ implies $[x\in f(R_N,p)$ if and only if $y\in f(R_N,p)]$.

Given that $(R_N,p)\in \mathcal{L}^N\times\Delta$ the alternatives $x,y\in X$, x is said to be terminally stochastically dominated by y if there exists $K\in\{1,\ldots,m-1\}$ such that $\Pr(\{rk(y)\leq K\}|(R_N,p)) > \Pr(\{rk(x)\leq K\}|(R_N,p))$ and $\Pr(\{rk(y)\leq r\}|(R_N,p)) \geq \Pr(\{rk(x)\leq r\}|(R_N,p))$ for all $r\in\{K+1,\ldots,m\}$.

An ECC f on $\mathcal{R}$ is said to satisfy No-Terminal Stochastic Domination if for all $(R_N,p)\in \mathcal{R}$ and $x,y\in X$: [x is terminally stochastically dominated by y] implies $[x\notin f(R_N,p)]$.

Lemma 1: If an ECC f on a domain $\mathcal{R}$ satisfies IIS and No-Terminal Stochastic Domination, then for all $(R_N,p)\in\mathcal{R}$ it must be the case that $f(R_N,p)\subset Mm(R_N,p)$.

Proof: Suppose f on a domain $\mathcal{R}$ satisfies No-Terminal Stochastic Domination.

Let $(R_N,p)\in\mathcal{R}$ and $x\in f(R_N,p)$.

By IIS, we may without loss of generality suppose that support (p) = N.

If worstrk $(x, (R_N,p)) = 1$, then Mm $(R_N,p) = \{x\}$, and so $x\in Mm(R_N,p)$. Hence suppose $r = \text{worstrk}(x, (R_N,p)) > 1$.

Towards a contradiction, suppose that for $y\in Mm(R_N,p)$, it is the case that $\text{worstrk}(y, (R_N,p)) = \rho < r$.

Thus, $1 = \Pr(\{rk(y)\leq\rho\}|(R_N,p)) > \Pr(\{rk(x)\leq\rho\}|(R_N,p))$ and $1 = \Pr(\{rk(y)\leq r\}|(R_N,p)) \geq \Pr(\{rk(x)\leq r\}|(R_N,p))$ for all $r\in\{\rho+1,\ldots,m\}$.

But then y terminally stochastically dominates x, contradicting our assumption that f satisfies No-Terminal Stochastic Domination.

Thus, $x\in Mm(R_N,p)$, and hence $f(R_N,p)\subset Mm(R_N,p)$. Q.E.D.

4. The problem with Max-min and a possible refinement

The problem with $Mm(R_N,p)$ and any ECC that does not discriminate between states of nature which have positive probability is that they might over-emphasize the "extremely unlikely" to absurd extents, thereby denying the decision-maker the right to exercise one's discretion within reasonable limits.

The following example comes from Lahiri (2022).

Example 1: X = {x,y}, n = 2,

$$(R_N,p) = \begin{array}{cc} \frac{1}{100} & \frac{99}{100} \\ \end{array} \begin{bmatrix} x & y \\ y & x \end{bmatrix}$$

$Mm(R_N,p) = \{x,y\}$. But, does 'x' have any reason to be treated at par with 'y', when there is a 99% chance that 'y' is going to be preferred to 'x'?

Hence, we consider the procedure below.

The next notation will prove useful in what follows.

Given that $(R_N,p) \in \mathcal{L}^N \times \Delta$ and $x \in X$, the probability of the worst rank of x at (R_N,p) denoted $Pr(WS(x,R_N,p)) = \sum_{i \in WS(x,R_N,p)} p_i$.

An ECC on $\mathcal{R}$ is called the probabilistic max-min choice correspondence, denoted f^{PMm}, if for all $(R_N,p) \in \mathcal{R}$, $f^{PMm}(R_N,p) = \{x \in Mm(R_N,p) | \ Pr(WS(x,R_N,p)) \leq Pr(WS(y,R_N,p))$ for all $y \in Mm(R_N,p)\}$, i.e. $f^{PMm}(R_N,p)$ is the set of max-min alternatives with the least total probability of securing the worst rank at (R_N,p).

Thus, an ECC is f^{PMm} which at any (R_N,p) in the domain of the ECC chooses those max-min alternatives whose max-min rank occurs with the least probability, i.e. the chosen alternative are those max-min alternatives each of which occurs at its worst rank with the least probability. In other words, f^{PMm} minimizes “the probability” with which a max-min rank occurs.

Clearly, $f^{PMm}(R_N,p)$ for Example 1 is {y}.

In view of the fact that domain $\mathcal{R}$ is a subset of $\mathcal{L}^N \times \Delta$, given any $(R_N,p) \in \mathcal{R}$, it is not possible for two different alternatives to have the same worst state of nature at (R_N,p)”.

Example 2: For m = 3 with X = {x,y,z} and n = 4, let (R_N,p) be defined as follows:

$$\begin{array}{cccc} \frac{2}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{array}$$
$$\begin{bmatrix} x & y & z & z \\ y & x & y & x \\ z & z & x & y \end{bmatrix}$$

Here $f^{PMm}(R_N,p) = \{x,y\}$, but y is terminally stochastically dominated by x.

Thus, we are led to the following refinement of the Probabilistic Max-min, and hence a further refinement of Max-min, which for each extended preference profile (R_N,p) beginning with $f^{PMm}(R_N,p)$ iteratively chooses (from among the chosen alternatives from the previous stage) those alternatives which occur at any rank with the highest (cumulative) probability of securing a better rank, all the way up to the second rank.

Let k = worstrk (x, R_N,p) for $x \in Mm(R_N,p)$.

If k = 1, then $Mm(R_N,p) = f^{PMm}(R_N,p)$ is a singleton.

If $f^{PMm}(R_N,p)$ is a singleton then STOP.

If $f^{PMm}(R_N,p)$ is not a singleton then k > 1.

(This does not necessarily mean that if k > 1, then $f^{PMm}(R_N,p)$ is not a singleton).

Let $f^{PMm}(R_N,p)$ be denoted k- $f^{PMm}(R_N,p)$, where k > 1 and the cardinality of the set k- $f^{PMm}(R_N,p)$ is greater than one.

Thus, k- $f^{PMm}(R_N,p) = \{x \in Mm(R_N,p) | \ Pr(\{rk(x) = k\}|(R_N,p)) \leq Pr(\{rk(y) = k\}|(R_N,p))$ for all $y \in Mm(R_N,p)\} = \{x \in Mm(R_N,p) | \ Pr(\{rk(x) \leq k\text{-}1\}|(R_N,p)) \geq Pr(\{rk(y) \leq k\text{-}1\}|(R_N,p))$ for all $y \in Mm(R_N,p)\}$

If k = 2, then $f^{PMm}(R_N,p)$ = 2- $f^{PMm}(R_N,p) = \{x \in Mm(R_N,p) | \Pr(\{rk(x) = 2\}|(R_N,p)) \leq \Pr(\{rk(y) = 2\}|(R_N,p))$ for all $y \in f^{PMm}(R_N,p)\} = \{x \in Mm(R_N,p) | \Pr(\{rk(x) = 1\}|(R_N,p)) \geq \Pr(\{rk(y) = 1\}|(R_N,p))$ for all $y \in f^{PMm}(R_N,p)\}$

Hence if k = 2, STOP.

Now suppose k > 2.

Having defined ρ- $f^{PMm}(R_N,p)$ with $k \geq \rho > 2$ and the cardinality of the set ρ- $f^{PMm}(R_N,p)$ greater than one, let (ρ-1)- $f^{PMm}(R_N,p) = \{x \in \rho$- $f^{PMm}(R_N,p) | \Pr(\{rk(x) \leq \rho - 2\}|(R_N,p)) \geq \Pr(\{rk(y) \leq \rho - 2\}|(R_N,p))$ for all $y \in \rho$- $f^{PMm}(R_N,p)\} = \{x \in \rho$- $f^{PMm}(R_N,p) | \Pr(\{rk(x) = \rho-1\}|(R_N,p)) \leq \Pr(\{rk(y) = \rho-1\}|(R_N,p))$ for all $y \in \rho$- $f^{PMm}(R_N,p)\}$.

Since this iterative process cannot go on indefinitely, we finally arrive at the set K^*- $f^{PMm}(R_N,p)$ with $K^* \geq 2$ such that:

either (i) cardinality of K- $f^{PMm}(R_N,p) > 1$ for all $K \in \{2,\ldots,k\}$, in which case $K^* = 2$;

or (ii) $K^* = \max \{K |$ K-$f^{PMm}(R_N,p)$ is a singleton$\}$.

Let APMm $(R_N,p) = K^*$- $f^{PMm}(R_N,p)$ if $x \in Mm(R_N,p)$ implies worstrk$(x, (R_N,p)) > 1$, and APMm $(R_N,p) = Mm(R_N,p)$, otherwise.

The extended choice correspondence f^{APMm} on $\mathcal{R}$, defined as $f^{APMm}(R_N,p)$ = APMm (R_N,p) for all $(R_N,p) \in \mathcal{R}$, is referred to as the Ascending Probabilistic Max-min Extended Choice Correspondence on $\mathcal{R}$.

In example 3, APMm $(R_N,p) = \{x\}$.

5. An axiomatic characterization of the Ascending Probabilistic Max-min ECC

Lemma 2: (a) The ECC f^{APMm} on $\mathcal{R}$ satisfies No-Terminal Stochastic Domination. (b) Let f on $\mathcal{R}$ be an ECC that satisfies IIS and No-Terminal Stochastic Domination. Then $f(R_N,p) \subset APMm(R_N,p)$ for all $(R_N,p) \in \mathcal{R}$.

Proof: Let $(R_N,p) \in \mathcal{R}$.

(a) By definition, $f^{APMm}(R_N,p) \subset f^{PMm}(R_N,p) \subset Mm(R_N,p)$. Let $x \in f^{APMm}(R_N,p)$ and suppose worstrk $(x,(R_N,p)) = K$. Towards a contradiction, suppose y terminally stochastically dominates x at (R_N,p).

If $y \notin Mm(R_N,p)$, then $1 = \Pr(\{rk(x) \leq K\}|(R_N,p)) > \Pr(\{rk(y) \leq K\}|(R_N,p))$ and $1 = \Pr(\{rk(x) \leq k\}|(R_N,p)) \geq \Pr(\{rk(y) \leq k\}|(R_N,p))$ for all $k \in \{K+1,\ldots,m\}$. This violates the requirement for y to terminally stochastically dominate x.

If $y \in Mm(R_N,p)$, then $0 < \Pr(\{rk(x) = K\}|(R_N,p)) \leq \Pr(\{rk(y) = K\}|(R_N,p))$ and $\Pr(\{rk(x) \leq k\}|(R_N,p)) = \Pr(\{rk(y) \leq k\}|(R_N,p)) = 1$ for all $k \in \{K+1,\ldots,m\}$, since $x \in f^{APMm}(R_N,p)$.

$0 < \Pr(\{rk(x) = K\}|(R_N,p)) \leq \Pr(\{rk(y) = K\}|(R_N,p))$ implies $\Pr(\{rk(x) \leq K\text{-}1\}|(R_N,p)) \geq \Pr(\{rk(y) \leq K\text{-}1\}|(R_N,p))$, since $\Pr(\{rk(x) \leq K\}|(R_N,p)) = 1 = \Pr(\{rk(y) \leq K\}|(R_N,p))$

Since y terminally stochastically dominates x, it must be the case that $\Pr(\{rk(y) \leq \rho\}|(R_N,p)) > \Pr(\{rk(x) \leq \rho\}|(R_N,p))$ for some $\rho < K\text{-}1$ and $\Pr(\{rk(y) \leq k\}|(R_N,p)) \geq \Pr(\{rk(x) \leq k\}|(R_N,p))$ for all $k > \rho$. This, in particular, implies $y \in f^{PMm}(R_N,p)$.

This contradicts our assumption that $x \in f^{APMm}(R_N,p)$.

Thus, f^{APMm} satisfies No-Terminal Stochastic Domination.

(b) Suppose f is an ECC on $\mathcal{R}$ that satisfies ISS and No-Terminal Stochastic Domination.

By lemma 1, for all $(R_N,p) \in \mathcal{R}$, it is the case that $f(R_N,p) \subset Mm(R_N,p)$.

Let $x \in f(R_N,p)$ and suppose worstrk $(x, (R_N,p)) = K$.

Since $x \in Mm(R_N,p)$, $\Pr(\{rk(x) \leq K\}|(R_N,p)) = 1$.

Towards a contradiction, suppose that there exists $y \in Mm(R_N,p)$ and $\rho \geq 1$ with $\rho \leq K\text{-}1$, satisfying $\Pr(\{rk(y) \leq \rho\}|(R_N,p)) > \Pr(\{rk(x) \leq \rho\}|(R_N,p))$ and $\Pr(\{rk(y) \leq k\}|(R_N,p)) \geq \Pr(\{rk(y) \leq k\}|(R_N,p))$ for all $k > \rho$.

This violates the requirement that f satisfies No-Terminal Stochastic Domination.

Thus, for $y \in Mm(R_N,p)$ and $\rho \geq 1$ with $\rho \leq K\text{-}1$, either (i) $\Pr(\{rk(y) \leq \rho\}|(R_N,p)) \leq \Pr(\{rk(x) \leq \rho\}|(R_N,p))$ in which case $x \in APMm(R_N,p)$, or (ii) $\Pr(\{rk(y) \leq \rho\}|(R_N,p)) > \Pr(\{rk(x) \leq \rho\}|(R_N,p))$, and for some $k > \rho$, in which case it must be that $\Pr(\{rk(y) \leq k\}|(R_N,p)) < \Pr(\{rk(y) \leq k\}|(R_N,p))$, the latter implying $y \notin APMm(R_N,p)$.

Since $APMm(R_N,p) \neq \phi$, (i) and (ii) imply that $x \in APMm(R_N,p)$.

Thus, $f(R_N,p) \subset APMm(R_N,p)$ as desired. Q.E.D.

Let $\mathcal{R}^{APMm} = \{(R_N,p) \in \mathcal{L}^N \times \Delta | APMm(R_N,p)$ is a singleton$\}$, i.e. the largest domain (of extended preference profiles) on which f^{PMm} is resolute.

An immediate and important consequence of Lemma 2 is the following Corollary.

Corollary to Lemma 2: A ECC f on $\mathcal{R}^{APMm}$ satisfies IIS and No-Terminal Stochastic Domination if and only if [f is resolute and $f = f^{APMm}$].

IIS and No-Terminal Stochastic Domination along with Probabilistic Neutrality can be used to establish an axiomatic characterization of the Ascending Probabilistic Max-min solution.

Proposition 1: An ECC f on $\mathcal{R}$ satisfies IIS, Probabilistic Neutrality and No-Terminal Stochastic Domination if and only if $f = f^{APMm}$.

Proof: It is easy to see that f^{APMm} on $\mathcal{R}$ satisfies Probabilistic Neutrality.

Hence, suppose that f on $\mathcal{R}$ satisfies IIS, Probabilistic Neutrality and No-Terminal Stochastic Domination.

Let $(R_N,p) \in \mathcal{R}$.

By Lemma 2, we know that $f(R_N,p) \subset APMm(R_N,p) = f^{APMm}(R_N,p)$.

If for $y \in Mm(R_N,p)$, the worst-rank $(y, (R_N,p)) = 1$, then $f^{APMm}(R_N,p)$ = Max-min Winners(M) is a singleton.

Thus, $f(R_N,p) = f^{APMm}(R_N,p)$.

Let $APMm(R_N,p) = K^*\text{-}f^{PMm}(R_N,p)$ with $K^* > 2$.

Then, $APMm(R_N,p) = K^*\text{-}f^{PMm}(R_N,p)$ is a singleton, so that $f(R_N,p) \subset APMm(R_N,p) = f^{APMm}(R_N,p)$ implies $f(R_N,p) = f^{APMm}(R_N,p)$.

Hence, suppose $K^* = 2$. Then for all $x,y \in APMm(R_N,p) = K^*\text{-}f^{PMm}(R_N,p)$, it is the case that $Pr(\{rk(y) = k\}|(R_N,p)) = Pr(\{rk(y) = k\}|(R_N,p))$ for all $k \in \{1,\ldots,m\}$.

By Probabilistic Neutrality, $x \in f(R_N,p)$ if and only if $y \in f(R_N,p)$.

Thus, $f(R_N,p) = f^{APMm}(R_N,p)$. Q.E.D.

Note: Given a strict ranking on X, the extended choice correspondence which for all $(R_N,p) \in \mathcal{R}$ chooses from $APMm(R_N,p)$, the alternative that is ranked best according to this strict ranking is singleton valued (resolute), satisfies IIS and No-Terminal Stochastic Domination, but does not satisfy Probabilistic Neutrality. The choice function that selects the whole set X for all $(R_N,p) \in \mathcal{R}$ satisfies IIS and Probabilistic Neutrality, but does not satisfy the No-Terminal Stochastic Domination.

6. Conclusion

The solution concept we suggest here violates the Condorcet consistency, which requires that if an alternative is preferred to all other alternatives in a pair-wise comparison, then such an alternative should be (the only alternative to be) chosen. This is easily seen for an extended preference profile with three alternatives: x, y, z, where x is ranked first with probability $\frac{501}{1000}$, and last with probability $\frac{499}{1000}$, whereas y is ranked second with probability 1. Our solution would select y, in spite of x being the Condorcet winner. The problem with x is its extreme volatility, and our solution concept protects the decision-maker from the not-unlikely adverse consequences that the choice of the Condorcet winner would expose him or her to.

An alternative way of proceeding with our analysis would be to use ordinal data matrices. An ordinal data matrix gives the probability with which each alternative is assigned each rank. Clearly, such a matrix is a bi-stochastic matrix of rational numbers, assuming that the probability with which each state of nature occurs is a rational number. The Birkhoff-von Neumann theorem states that every bi-stochastic matrix is an expected ranking matrix, though there may be more than one probability distribution over strict rankings that lead to the same expected ranking matrix. Many choice procedures based on preference profiles, including the procedure we discuss here, can be stated in terms of data contained in such matrices.

Aleskerov and Subochev (2013) study the representation of binary relations on finite sets by logical matrices. They are largely concerned with the "preferred with probability at least half" relation.

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The detectability of asymmetric distributions deviating from normality due to small skewness

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Abstract. The aim of this article is to test the ability of goodness-of-fit tests (GoFTs) to detect any deviations from normality. A very specific case is considered, namely the deviation from normality consisting in the coincidence of asymmetry and small γ_1 skewness. The first step in achieving the aforementioned aim is to compile a set of normality-oriented GoFTs commonly recommended for use, as described in the recently published literature. The second step is to create a family of asymmetric distributions with a non-constant γ_1, further referred to as alternatives. The formulas for calculating γ_1 are provided for each alternative. To compare the alternatives with the normal distribution, a relevant similarity measure is applied. The third step involves running a Monte Carlo simulation. The study investigates 21 GoFTs and 13 alternatives. The obtained results show that the $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ and H_n GoFTs prove most effective in detecting asymmetric distributions that deviate from normality due to small skewness, equal to even 0.05.

Keywords: normality, goodness-of-fit test, skewness
JEL: C1, C6

1. Introduction

Numerous goodness-of-fit tests (GoFTs) are discussed in the statistics-related literature. The most common normality test procedures available in statistical software are the Kolmogorov-Smirnov (KS) test (Kolmogorov, 1933; Smirnov, 1948), the Lilliefors (LF) test (Lilliefors, 1967), the Cramer-von Mises (CVM) test (Cramér, 1928), the Anderson-Darling (AD) test (Anderson & Darling, 1952), and the Shapiro-Wilk (SW) test (Shapiro & Wilk, 1965). The Power R package (Lafaye de Micheaux & Tran, 2016) from the R software proved the most useful to the research undertaken in this paper. The package offers a large set of generators of pseudo-random numbers that follow probability distributions which are used both frequently and sporadically. Moreover, the package provides many GoFTs for normality, uniformity and laplacity (see Section 4).

In the recent years, many articles have been devoted to GoFTs for normality, e.g.: Afeez et al. (2018), Ahmad and Khan (2015), Aliaga et al. (2003), Arnastauskaitė et al. (2021), Bayoud (2021), Bonett and Seier (2002), Bontemps and Meddahi (2005), Brys et al. (2008), Coin (2008), Desgagné et al. (2023), Desgagné and Lafaye de Micheaux (2018), Gel at el. (2007), Gel and Gastwirth (2008), Hernandez (2021),

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Kellner and Celisse (2019), Khatun (2021), Marange and Qin (2019), Mbah and Paothong (2015), Mishra et al. (2019), Nosakhare and Bright (2017), Noughabi and Arghami (2011), Razali and Wah (2011), Romao et al. (2010), Sulewski (2019), Sulewski (2022b), Tavakoli et al. (2019), Torabi et al. (2016), Uhm and Yi (2021), Uyanto (2022), Wijekularathna et al. (2020), Yap and Sim (2011), and Yazici and Yolacan (2007).

In this article, we focus on GoFTs for normality recommended for use when the alternatives are asymmetric, i.e. the skewness (γ_1) is non-zero (see Table 1). The results of applying the Monte Carlo method to assess the power of GoFTs are presented in Section 4.

Asymmetric distributions can be divided into distributions with constant γ_1 and non-constant γ_1. Distributions with constant γ_1 include exponential (EXP), Gumbel (GU), half-logistic (HL), half-Normal (HN), log-Weibull (LW) or extreme-value, or Maxwell (MX) and Rayleigh (Ry) distributions. Section 3 is devoted to distributions with non-constant γ_1.

The research results presented in the article by Sulewski (2022a) inspired further research and form the core of this article. That paper discusses the Easily Changeable Kurtosis (ECK) distribution. The ECK enables testing the ability of GoFTs to detect the deviations from normality by negative excess kurtosis $\bar{\gamma}_2$. The article shows that the most popular GoFTs do not distinguish the ECK distribution of negative $\bar{\gamma}_2$ (even $\bar{\gamma}_2 = -0.3$) from normal distribution. This is the case even when sample size $n = 30, 50$ and significance level $\alpha = 0.05$. The findings presented in the author's other works entitled 'Goodness-of-fit testing for normality when alternative distributions have undefined or constants skewness and kurtosis' and 'On the detectability of symmetric distributions that deviate from normality due to small excess kurtosis' (currently reviewed) also motivated further research in the discussed area. The common feature of this article and those mentioned in this paragraph is the testing of GoFTs.

The review of the recent statistics-related literature shows that $\gamma_1 \in [-0.25, 0.25]$ does not dominate in testing for normality. It is very interesting to see how the GoFT responds to samples coming from alternatives close to normal distribution. In this article, we will focus on γ_1 values close to zero. In other words, we use the values of alternative parameters to obtain the desired γ_1 values and similarity measure values of the alternatives to normal distribution.

The aim of this article is to test the ability of GoFTs to detect deviations from normality. A very specific case is considered, namely the deviation from normality consisting in the coincidence of asymmetry and small γ_1 values. The first step toward achieving the aforementioned aim is to compile a set of normality-oriented GoFTs commonly recommended for use, mainly on the basis of the review of recently-published source literature. The second step is to create a family of asymmetric distributions with non-constant γ_1, further referred to as alternatives. Formulas for calculating the γ_1 and $\bar{\gamma}_2$ values are provided for each distribution. In order to compare the alternatives with normal distribution, an appropriate similarity measure is applied. The third step involves performing a Monte Carlo simulation. The study is based on the use of 21 GoFTs and 13 alternatives.

The article is organised as follows: Section 2 presents 21 GoFTs for normality recommended in the literature as fit for use when the alternatives are asymmetric. Section 3 is devoted to the similarity measure of the normal distribution to the alternative distribution. Moreover, this part of the study presents asymmetric distribution with non-constant γ_1. Section 4 analyses the results of the Monte Carlo simulations. The summary and conclusions, compiled in Section 5, close the paper.

2. Goodness-of-fit tests for the Monte Carlo simulation

Hypothesis H_0 states that the data come from normal distribution. Hypothesis H_1 negates H_0. Table 1 presents the studied 21 GoFTs for normality (sorted by year) recommended in the literature in the recent years ($n \leq 100$) when alternatives are asymmetric. These GoFTs are used in the Monte Carlo simulations (see Section 4).

Table 1. GoFTs for normality when alternatives are asymmetric ($n \leq 100$)

GoFT	Recommended by
Anderson-Darling test (AD) (Anderson & Darling, 1952)	Afeez et al. (2018), Khatun (2021), Yap and Sim (2011)
Shapiro-Wilk test (SW) (Shapiro & Wilk, 1965)	Afeez et al. (2018), Bayoud (2021), Coin (2008), Hernandez (2021), Khatun (2021), Mishra et al. (2019), Romao et al. (2010), Wijekularathna et al. (2020), Yap and Sim (2011)
Kurtosis test (KT) (Shapiro et al., 1968)	Mishra et al. (2019)
D'Agostino skewness test (AS) (D'Agostino, 1970)	Mishra et al. (2019)
Shapiro-Francia test (SF) (Shapiro & Francia, 1972)	Khatun (2021), Nosakhare and Bright (2017)
D'Agostino-Pearson test (AP) (D'Agostino & Pearson, 1973)	Mishra et al. (2019)
Ryan-Joiner test (RJ) (Ryan & Joiner, 1976)	Nosakhare and Bright (2017)
T_{1n} test (T_{1n}) (LaRiccia, 1986)	Torabi et al. (2016)

Table 1. GoFTs for normality when alternatives are asymmetric ($n \leq 100$) (cont.)

GoFT	Recommended by
Jarque-Bera test (JB) (Jarque & Bera, 1987)	Brys et al. (2008), Yazici and Yolacan (2007)
1st Hosking test (H_1) (Hosking, 1990)	Arnastauskaitė et al. (2021)
1st Cabana-Cabana test (CC) (Cabaña & Cabaña, 1994)	Uyanto (2022)
Chen-Shapiro test (CS) (Chen & Shapiro, 1995)	Romao et al. (2010)
Adjusted Jarque-Bera test (AJB) (Urzua, 1996)	Nosakhare and Bright (2017)
ZA Zhang-Wu test (ZA) (Zhang & Wu, 2005)	Romao et al. (2010), Sulewski (2019), Uhm and Yi (2021), Uyanto (2022)
ZC Zhang-Wu test (ZC) (Zhang & Wu, 2005)	Romao et al. (2010), Uhm and Yi (2021)
β_3^2 Coin test (β_3^2), (Coin, 2008)	Coin (2008)
H_n test (H_n) Torabi et al. (2016)	Torabi et al. (2016)
X_{APD} test (X_{APD}) (Desgagné & Lafaye de Micheaux, 2018)	Desgagné et al. (2023)
B_v test (B_v) Tavakoli et al. (2019)	Tavakoli et al. (2019)
Modified Lilliefors test ($LF_{\overline{\alpha},\overline{\beta}}$) (Sulewski, 2022b)	Sulewski (2022b)
Delta test (δ) Bayoud (2021)	Bayoud (2021)

Source: author's work.

3. The similarity measure and the alternatives

3.1. Similarity measure

Let $f(x;\boldsymbol{\theta})$ be a probability density function (PDF) of an alternative distribution with vector of parameters $\boldsymbol{\theta}$. Similarity measure M of the alternative to the null distribution is defined as (Sulewski, 2022b)

$$M(\boldsymbol{\theta};\mu,\sigma) = \int_{-\infty}^{\infty} min[f(x;\boldsymbol{\theta}),\phi(x;\mu,\sigma)]dx, \quad (1)$$

where $\phi(x;\mu,\sigma)$ is the PDF of the normal distribution. $M(\boldsymbol{\theta};\mu,\sigma)$ takes the values of $[0,1]$ and equals 1 when the PDFs are identical. More details on distance and similarity measures can be found e.g. in Sulewski (2021).

3.2. Alternative distributions

Asymmetric alternatives with non-constant γ_1 used in Monte Carlo simulations can be divided into two groups. The first and second group includes monolithic and compound distributions, respectively, used in GoFTs for normality in recent articles. These alternatives are:

- Group I: beta (B), chi-squared (χ^2), gamma (G), generalised power (GP), inverse Gaussian (IG), lognormal (LOG), power normal (PN), SB Johnson (SB), Skew-flexible-normal (SFN), skew-normal (SN), SU Johnson (SU) and Weibull (W) distributions;
- Group II: location-contaminated normal (LCN), Gumbel-normal (GN), Laplace mixture (LM), Laplace-normal (LN), normal distribution with a plasticising component (NDPC), normal mixture (NM), plasticising component mixture (PCM), skew-normal mixture (SNM) and Weibull-normal (WN) distributions.

See Table 2 for more details. The distributions used in at least two articles (marked in bold) have been selected for the Monte Carlo simulation (see Section 4).

Table 2. Asymmetric alternatives (A) with non-constant γ_1 used in GoFTs for normality in the recent literature (in alphabetical order)

A	Article
B	Afeez et al. (2018), Arnastauskaitė et al. (2021), Bayoud (2021), Coin (2008), Desgagné and Lafaye de Micheaux (2018), Gel at el. (2007), Noughabi and Arghami (2011), Razali and Wah (2011), Romao et al. (2010), Sulewski (2019), Sulewski (2022b), Torabi et al. (2016), Uhm and Yi (2021), Uyanto (2022), Yap and Sim (2011), Yazici and Yolacan (2007)
χ^2	Arnastauskaitė et al. (2021), Bayoud (2021), Bontemps and Meddahi (2005), Coin (2008), Desgagné and Lafaye de Micheaux (2018), Nosakhare and Bright (2017), Razali and Wah (2011), Romao et al. (2010), Sulewski (2019), Sulewski (2022b), Tavakoli et al. (2019), Torabi et al. (2016), Uhm and Yi (2021), Wijekularathna et al. (2020)
G	Arnastauskaitė et al. (2021), Bayoud (2021), Desgagné and Lafaye de Micheaux (2018), Noughabi and Arghami (2011), Razali and Wah (2011), Romao et al. (2010), Tavakoli et al. (2019), Torabi et al. (2016), Uhm and Yi (2021), Uyanto (2022), Yap and Sim (2011), Yazici and Yolacan (2007)
GN	Sulewski (2022b)
GP	Desgagné et al. (2023), Desgagné and Lafaye de Micheaux (2018)
IG	Tavakoli et al. (2019)
LCN	Coin (2008), Yap and Sim (2011)
LM	Sulewski (2022b)
LN	Sulewski (2022b)
LOG	Arnastauskaitė et al. (2021), Bayoud (2021), Coin (2008), Desgagné and Lafaye de Micheaux (2018), Gel at el. (2007), Marange and Qin (2019), Noughabi and Arghami (2011), Romao et al. (2010), Sulewski (2019), Sulewski (2022b), Tavakoli et al. (2019), Torabi et al. (2016), Wijekularathna et al. (2020), Yap and Sim (2011), Yazici and Yolacan (2007)
NDPC	Sulewski (2022b)

Table 2. Asymmetric alternatives (A) with non-constant γ_1 used in GoFTs for normality in the recent literature (in alphabetical order) (cont.)

A	Article
NM	Romao et al. (2010), Sulewski (2022b)
PCM	Sulewski (2022b)
PN	Sulewski (2022b)
SB	Sulewski (2019), Sulewski (2022b), Torabi et al. (2016)
SFN	Sulewski (2022b)
SN	Bayoud (2021), Sulewski (2022b), Torabi et al. (2016), Uyanto (2022)
SU	Sulewski (2019), Torabi et al. (2016)
SNM	Sulewski (2022b)
W	Afeez et al. (2018), Ahmad and Khan (2015), Arnastauskaitė et al. (2021), Bayoud (2021), Coin (2008), Desgagné and Lafaye de Micheaux (2018), Nosakhare and Bright (2017), Noughabi and Arghami (2011), Romao et al. (2010), Sulewski (2022b), Tavakoli et al. (2019), Torabi et al. (2016), Uyanto (2022), Yap and Sim (2011), Yazici and Yolacan (2007)
WN	Sulewski (2022b)

Source: author's work.

The family of alternatives also includes two very interesting distributions ideally suited to the subject of this work, namely the Edgeworth series (ES) and the Pearson (P) distributions. Their parameters are γ_1 and $\bar{\gamma}_2$.

Let $\phi(x;0{,}1)$ and $\Phi(x;0{,}1)$ be the PDF and the cumulative density function (CDF) of the $N(0{,}1)$ distribution, respectively. Below, for the analysed alternatives, the PDF, the $M(\boldsymbol{\theta};0,\sigma)$ maximum value, and the $\gamma_1(\boldsymbol{\theta}), \bar{\gamma}_2(\boldsymbol{\theta}), \boldsymbol{\theta}(\gamma_1), \boldsymbol{\theta}(\bar{\gamma}_2)$ formulas are shown. The alternatives are presented in alphabetical order.

1. Beta distribution

$$f_B(x;a,b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)},\ x \in [0{,}1]\ (a > 0, b > 0)$$

$$M(11.372{,}11.372;0.5{,}0.105) = 0.990$$

$$\gamma_1(a,b) = \frac{2(b-a)\sqrt{a+b+1}}{(a+b+2)\sqrt{ab}}\,(\gamma_1 \in R),\ \gamma_1(a,b) = -\gamma_1(b,a)$$

$$\bar{\gamma}_2(a,b) = \frac{6[(a-b)^2(a+b+1) - ab(a+b+2)]}{ab(a+b+2)(a+b+3)}\ (\bar{\gamma}_2 \geq -2)$$

2. Chi-squared distribution

$$f_{\chi^2}(x;k)=\frac{x^{0.5k-1}exp(-0.5x)}{2^{0.5k}\Gamma(0.5k)},\ x\geq 0\ (k>0)$$

$$M(92.498;91.47,13.506)=0.973$$

$$\gamma_1(\mathrm{k})=\sqrt{\frac{8}{k}}\ (\gamma_1>0),\ \ k(\gamma_1)=\frac{8}{\gamma_1^2},\ \ \bar{\gamma}_2(k)=\frac{12}{k}\ (\bar{\gamma}_2>0),\ \ k(\bar{\gamma}_2)=\frac{12}{\bar{\gamma}_2}$$

3. Gamma distribution

$$f_{\boldsymbol{G}}(x;a,b)=\frac{x^{c-1}exp(-x/a)}{a^c\Gamma(c)},\ \ x\geq 0\ (a>0,b>0)$$

$$M(0.06,80.166;4.815,0.543)=0.979$$

$$\gamma_1(\mathrm{b})=\frac{2}{\sqrt{b}}\ (\gamma_1>0),b(\gamma_1)=\frac{4}{\gamma_1^2},\ \ \bar{\gamma}_2(b)=\frac{6}{b}\ (\bar{\gamma}_2>0),\ \ b(\bar{\gamma}_2)=\frac{6}{\bar{\gamma}_2}$$

4. Generalised power distribution (Komunjer, 2007)

Let $g(x;a,b)=2a^b(1-a)^b[a^b+(1-a)^b]^{-1}$, $(0<a<1,b>0)$, then

$$f_{\boldsymbol{GP}}(x;a,b)=\frac{g(x;a,b)^{\frac{1}{b}}}{\Gamma\left(1+\frac{1}{b}\right)}exp\left\{-\frac{g(x;a,b)}{\left[\frac{1}{2}+sgn(x)\left(\frac{1}{2}-a\right)\right]^b}|x|^b\right\},\ \ x\in R$$

$$M(0.5,2;0,0.707)=1.$$

Let $\alpha_k=\int_{-\infty}^{\infty}x^k f_{\boldsymbol{GP}}(x;a,b)$, then

$$\gamma_1(\mathrm{a,b})=\frac{\alpha_3-3\alpha_1\alpha_2+2\alpha_1^3}{(\alpha_2-\alpha_1^2)^{1.5}}\ (-10<\gamma_1<10),\ \ \gamma_1(\mathrm{a,b})=-\gamma_1(1-\mathrm{a,b})$$

$$\bar{\gamma}_2(a,b)=\frac{\alpha_4-4\alpha_1\alpha_3+6\alpha_1^2\alpha_2-3\alpha_1^4}{(\alpha_2-\alpha_1^2)^2}-3\ (\bar{\gamma}_2>-1.2).$$

5. Location contaminated normal distribution

$$f_{LCN}(x;a,w)=w\phi(x;a,1)+(1-w)\phi(x;0{,}1),\ x\in R\ (0\le w\le 1,\ a>0)$$

$$M(0{,}1;\ 0{,}1)=M(a,0;\ 0{,}1)=1$$

$$\gamma_1(a,w)=\frac{a^3w(2w^2-3w+1)}{(a^2w-a^2w^2+1)^{1.5}}\ (\gamma_1\in R)$$

$$\bar{\gamma}_2(a,w)=\frac{a^4w(-6w^3+12w^2-7w+1)}{(a^2w-a^2w^2+1)^2}\ (\bar{\gamma}_2\ge -2)$$

$$a(\bar{\gamma}_2,w)=\sqrt{\frac{\left(\frac{\bar{\gamma}_2}{6w^2-6w+1}+\sqrt{\frac{\bar{\gamma}_2}{12w^3-6w^4-7w^2+w}}\right)(6w^2-6w+1)}{\bar{\gamma}_2w^2-6w+6w^2-\bar{\gamma}_2w+1}}$$

$$w(\bar{\gamma}_2,a)=\frac{a+\sqrt{\frac{4\bar{\gamma}_2+\bar{\gamma}_2a^2+4a^2+2\sqrt{a^4-4a^2\bar{\gamma}_2-24\bar{\gamma}_2}}{\bar{\gamma}_2+6}}}{2a}$$

6. Lognormal distribution

$$f_{LOG}(x;a,b)=\frac{1}{xb\sqrt{2\pi}}exp\left[-\frac{1}{2}\left(\frac{ln(x)-a}{b}\right)^2\right],\ x>0\ (a\in R,\ b>0)$$

$$M(0.103{,}0.096;\ 1.106{,}0.106)=0.974$$

$$\gamma_1(b)=[exp(b^2)+2]\sqrt{exp(b^2)-1}\ (\gamma_1\ge 0)$$

$$\bar{\gamma}_2(b)=exp(4b^2)+2exp(3b^2)+3exp(2b^2)-6\ (\bar{\gamma}_2\ge 0)$$

7. Normal mixture distribution

$$f_{NM}(x;\ a,b,w)=w\phi(x;\ 0{,}1)+(1-w)\phi(x;\ a,b),$$
$$x\in R\ (a\in R,\ 0\le w\le 1,\ b>0)$$

$$M(a,b,1;\ 0{,}1)=M(0{,}1{,}0;\ 0{,}1)=M(0{,}1,\omega;\ 0{,}1)=1$$

$$\gamma_1(a,b,w)=\frac{-w(2w^2-3w+1)a^3+aw(3w-3b^2w+3b^2-3)}{[w(a^2-b^2-a^2w+1)+b^2]^{1.5}}\ (\gamma_1\in R)$$

$$\gamma_1(a,b,w)=-\gamma_1(-a,b,w)$$

$$\bar{\gamma}_2(a,b,w)\frac{(6w^2-6w+1)a^4+a^2(12b^2w-12w-6b^2+6)+3b^4-6b^2+3}{(w-w^2)^{-1}[w(a^2-b^2-a^2w+1)+b^2]^2}(\bar{\gamma}_2\ge -2)$$

8. SB distribution (Johnson, 1949)

$$f_{SB}(x;a,b) = \frac{b}{x(1-x)}\phi\left[a + bln\left(\frac{x}{1-x}\right);0,1\right],\ x \in [0,1]\ (a \in R,\ b > 0)$$

$$M(0,2.669;\ 0.5,0.093) = 0.999$$

Let $\alpha_k = \int_0^1 x^k f_{\boldsymbol{SB}}(x;a,b)$, then

$$\gamma_1(\mathrm{a,b}) = \frac{\alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3}{(\alpha_2 - \alpha_1^2)^{1.5}}(\gamma_1 \in R),\ \ \gamma_1(\mathrm{a,b}) = -\gamma_1(-\mathrm{a,b})$$

$$\bar{\gamma}_2(a,b) = \frac{\alpha_4 - 4\alpha_1\alpha_3 + 6\alpha_1^2\alpha_2 - 3\alpha_1^4}{(\alpha_2 - \alpha_1^2)^2} - 3\ (\bar{\gamma}_2 \geq -2)$$

9. Skew-normal distribution (Azzalini, 1985)

$$f_{SN}(x;a) = 2\phi(x;0,1)\ \Phi(ax;0,1),\ \ x \in R\ (a \in R)$$

$$M(0;\ 0,1) = 1$$

$$\gamma_1(\mathrm{a}) = \frac{a^3\sqrt{2}(4-\pi)}{(\pi - 2a^2 + \pi a^2)^{1.5}}\ (-1 < \gamma_1 < 1),\ \gamma_1(\mathrm{a}) = -\gamma_1(-\mathrm{a})$$

$$\bar{\gamma}_2(a) = \frac{4a^4(2\pi - 6)}{(\pi - 2a^2 + \pi a^2)^2}\ (0 \leq \bar{\gamma}_2 \leq 0.869)$$

$$a(\bar{\gamma}_2) = \pm\sqrt{\frac{\pi\left(2\bar{\gamma}_2\sqrt{\pi-3} + 6\sqrt{2\bar{\gamma}_2} - \pi\bar{\gamma}_2\sqrt{\pi-3} - 2\pi\sqrt{2\bar{\gamma}_2}\right)}{\sqrt{\pi-3}(4\bar{\gamma}_2 - 8\pi - 4\pi\bar{\gamma}_2 + \pi^2\bar{\gamma}_2 + 24)}}$$

10. SU distribution (Johnson, 1949)

$$f_{SU}(x;b,c,d) = \frac{d}{\sqrt{x^2+b^2}}\phi\left[c + dsinh^{-1}\left(\frac{x}{b}\right);0,1\right],\ \ x \in R\ (b > 0, c \in R, d > 0)$$

$$M(1.375,0,11.129;\ 0,0.124) = 0.998$$

Let

$$W = exp(d^{-2}),\ \ K_1 = W^2(W^4 + 2W^3 + 3W^2 - 3)cosh\left(\frac{4c}{d}\right),$$

$$K_2 = 4W^2(W+2)cosh\left(\frac{2c}{d}\right),\ V = \frac{b^2}{2}(W-1)\left[Wcosh\left(\frac{2c}{d}\right) + 1\right],$$

then

$$\gamma_1(\mathrm{c,d}) = \frac{-b^3\sqrt{W}(W-1)^2\left[W(W+2)sinh\left(\frac{3c}{d}\right) + 3sinh\left(\frac{c}{d}\right)\right]}{4V^{1.5}}\ (\gamma_1 \in R)$$

$$\gamma_1(\mathrm{c,d}) = -\gamma_1(-\mathrm{c,d})$$

$$\bar{\gamma}_2(c,d) = \frac{b^4(W-1)^2[K_1 + K_2 + 6W + 3]}{8V^2} - 3\ (\bar{\gamma}_2 \geq 2).$$

11. Weibull distribution (Weibull, 1951)

$$f_W(x;a,b) = \frac{b}{a^b}x^{b-1}\exp\left[-\left(\frac{x}{a}\right)^b\right],\ x \geq 0\ (a > 0,\ b > 0)$$

$$M(1.851{,}3.603;\ 1.673{,}0.532) = 0.985$$

Let $\Gamma_k = \Gamma(1 + k/b)$, then

$$\gamma_1(b) = \frac{2\Gamma_1^3 - 3\Gamma_1\Gamma_2 + \Gamma_3}{(\Gamma_2 - \Gamma_1^2)^{1.5}}\ (\gamma_1 \geq -1.14)$$

$$\bar{\gamma}_2(b) = \frac{\Gamma_4 - 3\Gamma_2^2 - 4\Gamma_1\Gamma_3 + 12\Gamma_1^2\Gamma_2 - 6\Gamma_1^4}{(\Gamma_2 - \Gamma_1^2)^2}\ (\bar{\gamma}_2 \geq -0.289).$$

12. Edgeworth series distribution (Aliaga et al., 2003)

$$f_{ES}(x;\ \gamma_{1,}, \bar{\gamma}_2) = \frac{\phi(x;0{,}1)}{\left[1 + \frac{1}{3!}\gamma_{1,}(x^3 - 3x) + \frac{1}{4!}\bar{\gamma}_2(x^4 - 6x^2 + 3)\right]^{-1}},$$
$$x \in R\ (\gamma_1 \in R, \bar{\gamma}_2 \geq -2)$$

$$M(0{,}0;\ 0{,}1) = 1$$

The PDF formula is introduced in the Appendix.

13. Pearson distribution (Pearson, 1916)

Let

$$a = \frac{2\bar{\gamma}_2 - 3\gamma_1^2}{10\bar{\gamma}_2 - 5\gamma_1^2 + 12},\ \ b = \frac{|\gamma_1|(\bar{\gamma}_2 + 6)}{10\bar{\gamma}_2 - 5\gamma_1^2 + 12}, c = \frac{4\bar{\gamma}_2 - 3\gamma_1^2 + 12}{10\bar{\gamma}_2 - 5\gamma_1^2 + 12}, \tag{2}$$
$$\Delta = b^2 - 4ac$$

then

$$f_P(x;\gamma_1,\bar{\gamma}_2) = \begin{cases} \dfrac{exp\left[\dfrac{2ab-b}{a(2ax+b)}\right]}{C_2(2ax+b)^{1/a}} & \Delta = 0 \\ \dfrac{exp\left[\dfrac{b-2ab}{a\sqrt{4ac-b^2}}tan^{-1}\left(\dfrac{2ax+b}{\sqrt{4ac-b^2}}\right)\right]}{C_4(ax^2+bx+c)^{1/(2a)}} & \Delta < 0 \\ \dfrac{\left(\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right)^{\frac{b-2ab}{2a\sqrt{b^2-4ac}}}}{C_8(ax^2+bx+c)^{1/(2a)}} & \Delta > 0, \end{cases}$$

where C_2, C_4, C_8 are normalising constants given by

$$C_2 = \int_{-\infty}^{\infty} \frac{exp\left[\dfrac{2ab-b}{a(2ax+b)}\right]}{(2ax+b)^{1/a}}dx, \tag{2}$$

$$C_4 = \int_{-\infty}^{\infty} \frac{exp\left[\dfrac{b-2ab}{a\sqrt{4ac-b^2}}tan^{-1}\left(\dfrac{2ax+b}{\sqrt{4ac-b^2}}\right)\right]}{(ax^2+bx+c)^{1/(2a)}}dx, \tag{3}$$

$$C_8 = \int_{-\infty}^{\infty} \frac{\left(\dfrac{2ax+b-\sqrt{\Delta}}{2ax+b+\sqrt{\Delta}}\right)^{\frac{b-2ab}{2a\sqrt{\Delta}}}}{C_8(ax^2+bx+c)^{1/(2a)}}dx, \tag{4}$$

$$M(0,0;\ 0,1) = 1.$$

The PDF formula is introduced in the Appendix.

The 13 above-mentioned distributions are grouped in Table 3 according to different properties. Table 3 shows that most of the analysed distributions have an infinite domain and assume negative or positive skewness values. The normal distribution is a special case of six distributions.

Table 3. Asymmetric distributions with non-constant $\bar{\gamma}_2$ grouped by different properties. The numbers of distributions with the given properties are provided in brackets

Property	Distributions
Finite domain	B, SB (2)
Infinite domain	$\chi^2, G, GP, LCN, LOG, NM, SN, SU, W, ES, P$ (11)
$M(\mathbf{\theta}; \mu, \sigma) = 1$ for some $\mathbf{\theta}, \mu, \sigma$	GP, LCN, NM, SN, ES, P (6)
$\gamma_1 < 0$	(0)
$\gamma_1 > 0$	LOG, G, χ^2 (3)
$\gamma_1 < 0 \lor \gamma_1 > 0$	$B, GP, LCN, NM, SB, SN, SU, W, ES, P$ (10)
Unimodal	$B, \chi^2, G, GP, LOG, SB, SN, SU, W, ES, P$ (11)
Bimodal	LCN, NM (2)

Source: author's work.

4. Monte Carlo simulation

For alternatives numbered from 1 to 13 (see Section 3.2), 21 large-scale experiments are performed, each dedicated to one of the GoFTs (see Section 2). Each experiment involves generating 10^4 samples of size $n = 25$. The samples come from a given alternative. Each sample is tested for normality at significance level $\alpha = 0.05$. The values of the alternative parameters are determined to obtain appropriate γ_1 values. The power of tests (PoTs) are calculated for the given γ_1 values.

All calculations are performed in R software using the codes presented in Table 4. A research tool facilitating the Monte Carlo power simulation studies for GoFTs in R called the PowerR package (Lafaye de Micheaux & Tran, 2016) proved very helpful in the process. The 'statcompute()' function calculates the test statistic value and the p-value for the GoFT described by the 'stat.index' argument, the sample described by the argument as 'data', and the significance level described by the argument as 'level'. Thus, e.g. in the case of the ZA test, the calculations take the following form: statcompute(stat.index = 4, data = sample, level = 0.05). See Table 4 for more information.

Table 5 presents the generator formulas for all the alternatives described in Section 3.

Table 4. The R codes of the used GoFTs

GoFT	R codes	GoFT	R codes
AD	ad.test	CS	statcompute(stat.index = 26…)
SW	shapiro.test	AJB	ajb.norm.test
KT	kurtosis.norm.test	ZA	statcompute(stat.index = 4…)
AS	agostino.test	ZC	statcompute(stat.index = 3…)
SF	sf.test	β_3^2	statcompute(stat.index = 30…)
AP	dagoTest	H_n	author's function, see Appendix
RJ	author's function, see Appendix	X_{APD}	statcompute(stat.index = 36…)
T_{1n}	author's function, see Appendix	B_v	author's function, see Appendix
JB	jarque.test	$LF_{\overline{\alpha},\overline{\beta}}$	author's function, see Appendix
H1	statcompute(stat.index = 10…)	δ	author's function, see Appendix
CC	statcompute(stat.index = 19…)		

Source: author's work.

Table 5. Generator formulas for the analysed alternatives (A) in R

A	Generator	A	Generator
B	rbeta(n,a,b)	SB	rJohnsonSB(n,a,b,0,1)
χ^2	rchisq(n,k)	SN	rskewnorm(n,0,1,a)
G	rgamma(n,b,1/a)	SU	rJohnsonSU(n,c,d,0,b))
GP	rGP(n,a,b) see Appendix	W	rweibull(n,b,1.851)
LCN	rLCN(n,a,b) see Appendix	ES[a]	$rEdge(n,\gamma_{1,},\overline{\gamma}_2,x_l,x_u)$, see Appendix
LOG	rlnorm(n,0.103,b)	P	mom = c(0,1, $\gamma_{1,}$, $\overline{\gamma}_2$)
NM	rNM(n,a,b,ω) see Appendix		rpearson(n,moments=mom))

a The quality of built-in function rCornishFisher(n,1, $\gamma_{1,}$, $\overline{\gamma}_2$) was not satisfactory.
Source: author's work.

The simulation results for the alternatives are presented in alphabetical order in Tables 6–22. We assume that a GoFT detects negative or positive γ_1 if its power reaches at least 0.06. PoT values are marked in bold, while the highest average PoT values for positive and negative γ_1 are underlined.

Table 6. $B(a,b)$ distribution. PoT versus γ_1 and $M(a,b;0.5,0.105)$ for $n=25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	–0.312	–0.306	–0.270	–0.338	–0.242	–0.233	–0.242	–0.338	–0.270	–0.306	-0.312
a	7.438	8.135	9.824	7.733	11.372	11.372	10.050	6.294	7.040	5.426	4.596
b	4.595	5.426	7.040	6.294	10.050	11.372	11.372	7.733	9.824	8.135	7.438
M	0.600	0.650	0.700	0.800	0.881	0.990	0.881	0.800	0.700	0.650	0.600
GoFT	PoT										
AD	0.058	0.052	0.050	0.044	0.045	0.042	0.046	0.045	0.046	0.052	**0.060**
SW	0.058	0.050	0.046	0.041	0.039	0.039	0.041	0.039	0.045	0.050	0.060
KT	0.038	0.032	0.036	0.029	0.032	0.033	0.032	0.031	0.033	0.033	0.038
AS	0.041	0.035	0.035	0.026	0.029	0.029	0.028	0.025	0.035	0.035	0.045
SF	0.046	0.038	0.038	0.031	0.034	0.034	0.035	0.030	0.037	0.042	0.049
AP	0.041	0.034	0.036	0.027	0.030	0.031	0.030	0.029	0.035	0.036	0.044
RJ	0.043	0.035	0.035	0.028	0.032	0.031	0.032	0.028	0.033	0.039	0.045
T_1	0.057	0.051	0.048	0.035	0.037	0.035	0.038	0.036	0.043	0.048	0.055
JB	0.034	0.027	0.030	0.022	0.027	0.026	0.025	0.022	0.029	0.029	0.038
H1	0.053	0.047	0.043	0.036	0.040	0.041	0.040	0.039	0.045	0.048	0.052
CC	0.040	0.035	0.035	0.026	0.030	0.030	0.030	0.027	0.035	0.039	0.048
CS	**0.060**	0.052	0.048	0.043	0.040	0.041	0.042	0.042	0.046	0.051	**0.061**
AJB	0.032	0.024	0.028	0.020	0.025	0.026	0.024	0.020	0.028	0.026	0.035
ZA	0.054	0.044	0.044	0.036	0.036	0.036	0.034	0.035	0.041	0.046	0.056
ZC	0.053	0.045	0.043	0.038	0.036	0.037	0.038	0.035	0.041	0.048	0.058
β_3^2	0.039	0.039	0.042	0.038	0.042	0.041	0.041	0.041	0.043	0.038	0.042
H_n	0.050	0.047	0.051	0.047	0.050	0.054	**0.062**	**0.068**	**0.071**	**0.081**	**0.093**
X_{APD}	0.051	0.043	0.043	0.035	0.037	0.036	0.038	0.036	0.040	0.044	0.049
B_v	**0.071**	**0.061**	**0.062**	0.054	0.051	0.049	0.052	0.055	0.060	**0.061**	**0.070**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.083**	**0.070**	**0.065**	0.054	0.053	0.045	0.053	0.058	**0.063**	**0.074**	**0.081**
δ	0.042	0.041	0.041	0.038	0.041	0.042	0.050	0.052	0.055	**0.066**	**0.072**

Source: author’s work.

Table 7. $\chi^2(k)$ distribution. PoT versus γ_1 and $M(k;\mu,\sigma)$ for $n = 25$

γ_1	0.294	0.350	0.400	0.450	0.500	0.550	0.600	0.650	0.700	0.705	0.800
$\bar{\gamma}_2$	0.130	0.184	0.240	0.304	0.375	0.454	0.540	0.634	0.735	0.844	0.960
0k	92.550	65.310	50.000	39.510	32.000	26.450	22.220	18.930	16.330	14.220	12.500
μ	91.410	66.930	51.864	42.556	36.448	31.503	23.608	20.842	23.742	15.101	21.708
σ	13.503	12.750	12.639	11.790	10.592	10.663	13.538	13.895	10.950	15.337	12.699
M	0.973	0.9010	0.850	0.801	0.750	0.700	0.650	0.602	0.550	0.511	0.450
GoFT	PoT										
AD	**0.068**	**0.076**	**0.087**	**0.095**	**0.098**	**0.122**	**0.144**	**0.150**	**0.164**	**0.188**	**0.212**
SW	**0.072**	**0.087**	**0.099**	**0.109**	**0.121**	**0.143**	**0.167**	**0.181**	**0.203**	**0.221**	**0.256**
KT	**0.065**	**0.074**	**0.075**	**0.083**	**0.088**	**0.098**	**0.116**	**0.117**	**0.130**	**0.133**	**0.153**
AS	**0.080**	**0.095**	**0.104**	**0.117**	**0.133**	**0.152**	**0.179**	**0.191**	**0.209**	**0.233**	**0.259**
SF	**0.078**	**0.090**	**0.098**	**0.110**	**0.122**	**0.145**	**0.170**	**0.179**	**0.199**	**0.218**	**0.248**
AP	**0.074**	**0.088**	**0.092**	**0.106**	**0.113**	**0.129**	**0.152**	**0.158**	**0.174**	**0.192**	**0.213**
RJ	**0.073**	**0.085**	**0.094**	**0.106**	**0.114**	**0.140**	**0.163**	**0.172**	**0.191**	**0.209**	**0.237**
T_1	**0.081**	**0.091**	**0.106**	**0.120**	**0.138**	**0.157**	**0.187**	**0.201**	**0.223**	**0.253**	**0.291**
JB	**0.076**	**0.090**	**0.094**	**0.108**	**0.114**	**0.134**	**0.157**	**0.164**	**0.180**	**0.202**	**0.225**
H1	**0.070**	**0.079**	**0.090**	**0.100**	**0.106**	**0.124**	**0.150**	**0.154**	**0.180**	**0.192**	**0.216**
CC	**0.080**	**0.095**	**0.102**	**0.120**	**0.134**	**0.155**	**0.182**	**0.195**	**0.216**	**0.239**	**0.268**
CS	**0.072**	**0.085**	**0.097**	**0.107**	**0.120**	**0.141**	**0.164**	**0.179**	**0.202**	**0.219**	**0.255**
AJB	**0.074**	**0.086**	**0.090**	**0.103**	**0.108**	**0.126**	**0.147**	**0.152**	**0.166**	**0.186**	**0.205**
ZA	**0.074**	**0.088**	**0.101**	**0.110**	**0.124**	**0.150**	**0.174**	**0.188**	**0.211**	**0.233**	**0.265**
ZC	**0.076**	**0.088**	**0.097**	**0.108**	**0.119**	**0.146**	**0.167**	**0.182**	**0.205**	**0.222**	**0.254**
β_3^2	0.052	0.055	0.058	0.058	0.058	0.058	**0.064**	**0.062**	**0.064**	**0.066**	**0.073**
H_n	**0.096**	**0.106**	**0.120**	**0.127**	**0.139**	**0.166**	**0.189**	**0.204**	**0.217**	**0.251**	**0.274**
X_{APD}	**0.071**	**0.077**	**0.087**	**0.099**	**0.108**	**0.124**	**0.146**	**0.154**	**0.172**	**0.191**	**0.215**
B_v	**0.064**	**0.074**	**0.083**	**0.083**	**0.098**	**0.110**	**0.128**	**0.140**	**0.158**	**0.176**	**0.200**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.084**	**0.096**	**0.104**	**0.122**	**0.126**	**0.148**	**0.165**	**0.172**	**0.180**	**0.206**	**0.222**
δ	**0.085**	**0.097**	**0.110**	**0.119**	**0.131**	**0.154**	**0.179**	**0.191**	**0.206**	**0.230**	**0.262**

Source: author's work.

Table 8. $\mathrm{G}(a,b)$ distribution. PoT versus γ_1 and $M(a,b;4.815,0.543)$ for $n=25$

γ_1	0.223	0.300	0.350	0.400	0.450	0.500	0.550	0.600	0.650	0.700	0.750
$\bar{\gamma}_2$	0.075	0.135	0.184	0.240	0.304	0.375	0.454	0.540	0.634	0.735	0.844
a	0.060	0.107	0.140	0.182	0.228	0.277	0.325	0.410	0.452	0.577	0.590
b	80.436	44.444	32.653	25.000	19.753	16.000	13.223	11.111	9.467	8.163	7.111
M	0.979	0.850	0.750	0.700	0.650	0.600	**0.550**	**0.550**	**0.500**	**0.500**	0.450
GoFT	PoT										
AD	**0.063**	**0.069**	**0.077**	**0.083**	**0.101**	**0.107**	**0.120**	**0.135**	**0.153**	**0.171**	**0.187**
SW	**0.065**	**0.076**	**0.086**	**0.095**	**0.115**	**0.129**	**0.141**	**0.166**	**0.181**	**0.204**	**0.226**
KT	**0.061**	**0.063**	**0.068**	**0.079**	**0.085**	**0.097**	**0.097**	**0.111**	**0.115**	**0.129**	**0.136**
AS	**0.066**	**0.077**	**0.093**	**0.105**	**0.121**	**0.142**	**0.152**	**0.172**	**0.192**	**0.214**	**0.234**
SF	**0.069**	**0.077**	**0.087**	**0.097**	**0.117**	**0.129**	**0.144**	**0.161**	**0.180**	**0.201**	**0.221**
AP	**0.065**	**0.071**	**0.081**	**0.093**	**0.105**	**0.123**	**0.133**	**0.149**	**0.157**	**0.177**	**0.192**
RJ	**0.066**	**0.073**	**0.082**	**0.092**	**0.110**	**0.123**	**0.137**	**0.155**	**0.174**	**0.193**	**0.212**
T_1	**0.066**	**0.080**	**0.095**	**0.108**	**0.119**	**0.143**	**0.157**	**0.184**	**0.204**	**0.229**	**0.256**
JB	**0.066**	**0.073**	**0.083**	**0.094**	**0.109**	**0.127**	**0.136**	**0.153**	**0.164**	**0.184**	**0.200**
H1	**0.065**	**0.070**	**0.077**	**0.088**	**0.103**	**0.114**	**0.128**	**0.144**	**0.159**	**0.178**	**0.192**
CC	**0.067**	**0.077**	**0.094**	**0.106**	**0.122**	**0.143**	**0.156**	**0.177**	**0.199**	**0.219**	**0.241**
CS	**0.064**	**0.073**	**0.084**	**0.093**	**0.113**	**0.127**	**0.140**	**0.163**	**0.179**	**0.201**	**0.223**
AJB	**0.065**	**0.070**	**0.080**	**0.089**	**0.102**	**0.120**	**0.128**	**0.143**	**0.152**	**0.172**	**0.184**
ZA	**0.065**	**0.077**	**0.088**	**0.097**	**0.115**	**0.132**	**0.147**	**0.167**	**0.189**	**0.214**	**0.237**
ZC	**0.067**	**0.075**	**0.084**	**0.095**	**0.113**	**0.131**	**0.144**	**0.163**	**0.182**	**0.205**	**0.226**
β_3^2	0.055	0.051	0.053	0.057	0.059	**0.062**	**0.061**	**0.064**	**0.060**	**0.066**	**0.065**
H_n	**0.084**	**0.098**	**0.106**	**0.120**	**0.136**	**0.148**	**0.160**	**0.182**	**0.205**	**0.226**	**0.245**
X_{APD}	**0.064**	**0.068**	**0.075**	**0.088**	**0.101**	**0.113**	**0.123**	**0.142**	**0.154**	**0.175**	**0.187**
B_v	**0.062**	**0.068**	**0.069**	**0.079**	**0.095**	**0.102**	**0.116**	**0.131**	**0.145**	**0.159**	**0.178**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.076**	**0.086**	**0.101**	**0.108**	**0.121**	**0.136**	**0.136**	**0.153**	**0.178**	**0.187**	**0.202**
δ	**0.077**	**0.088**	**0.095**	**0.109**	**0.125**	**0.138**	**0.152**	**0.174**	**0.194**	**0.210**	**0.234**

Source: author's work.

Table 9. $\mathrm{GP}(a,2)$ distribution. PoT versus γ_1 and $M(a,2;0{,}0.707)$ for $n=25$

γ_1	−0.250	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.045	0.029	0.016	0.007	0.002	0	0.002	0.007	0.016	0.029	0.045
a	0.579	0.563	0.547	0.531	0.516	0.5	0.484	0.469	0.453	0.437	0.421
M	0.916	0.934	0.951	0.968	0.984	1	0.984	0.968	0.951	0.934	0.916
GoFT	PoT										
AD	**0.067**	**0.065**	0.057	0.057	0.049	0.051	0.051	0.050	0.058	0.060	**0.069**
SW	**0.068**	**0.067**	0.058	0.057	0.048	0.053	0.053	0.052	0.060	**0.062**	**0.067**
KT	0.057	0.056	0.054	0.054	0.048	0.051	0.052	0.051	0.053	0.058	0.054
AS	**0.068**	**0.062**	0.058	0.055	0.046	0.052	0.055	0.053	0.059	**0.063**	**0.065**
SF	**0.068**	**0.067**	**0.062**	0.057	0.049	0.056	0.055	0.051	0.060	**0.063**	**0.067**
AP	**0.064**	0.059	0.056	0.053	0.048	0.052	0.053	0.051	0.055	**0.064**	0.059
RJ	**0.065**	**0.061**	0.058	0.052	0.046	0.053	0.052	0.049	0.056	0.060	**0.064**
T_1	**0.070**	**0.065**	**0.060**	0.056	0.048	0.049	0.053	0.052	**0.060**	**0.064**	**0.068**
JB	**0.064**	0.058	0.055	0.053	0.047	0.051	0.054	0.050	0.055	**0.062**	**0.060**
H1	**0.065**	**0.062**	0.058	0.058	0.050	0.056	0.053	0.051	0.059	0.058	**0.065**
CC	**0.067**	**0.062**	0.057	0.055	0.047	0.050	0.055	0.050	0.059	**0.064**	**0.067**
CS	**0.066**	**0.066**	0.059	0.056	0.048	0.053	0.051	0.052	0.059	**0.062**	**0.067**
AJB	0.060	0.059	0.053	0.054	0.045	0.051	0.054	0.050	0.054	**0.061**	0.058
ZA	**0.067**	**0.066**	0.058	0.055	0.047	0.053	0.053	0.053	0.057	**0.062**	**0.065**
ZC	**0.067**	**0.063**	**0.060**	0.055	0.048	0.053	0.051	0.053	0.058	**0.063**	**0.064**
β_3^2	0.049	0.050	0.048	0.050	0.049	0.053	0.053	0.049	0.051	0.052	0.055
H_n	0.052	0.052	0.054	0.056	0.054	0.057	0.059	**0.066**	**0.079**	**0.084**	**0.096**
X_{APD}	**0.064**	0.060	0.057	0.056	0.047	0.054	0.050	0.051	0.056	**0.061**	**0.063**
B_v	0.059	**0.063**	0.059	0.053	0.050	0.056	0.050	0.053	0.055	**0.061**	**0.068**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.085**	**0.084**	**0.069**	**0.066**	0.059	0.049	0.055	0.060	**0.069**	**0.077**	**0.086**
δ	0.048	0.049	0.048	0.051	0.046	0.051	0.054	0.055	**0.069**	**0.073**	**0.086**

Source: author's work.

Table 10. $\mathrm{LCN}(a,\omega)$ distribution. PoT versus γ_1 and $M(a,\omega;\mu,\sigma)$ for $n=25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.110	0.110	0.065	0.020	–0.048	0	–0.138	–0.116	–0.029	0.121	0.124
a	1.630	1.492	1.316	1.108	0.977	0	1.240	1.307	1.340	1.501	1.633
ω	0.845	0.866	0.855	0.823	0.660	1	0.413	0.337	0.237	0.127	0.147
μ	0.884	0.888	0.902	0.902	0.754	0	0.572	0.562	0.544	0.729	0.760
σ	0.990	0.976	0.915	0.915	1.155	1	1.218	1.320	1.393	1.332	1.516
M	0.786	0.821	0.868	0.917	0.961	1	0.971	0.930	0.880	0.800	0.791
GoFT	PoT										
AD	**0.069**	**0.061**	0.056	0.053	0.054	0.052	0.050	0.050	0.055	**0.061**	**0.065**
SW	**0.072**	**0.068**	**0.061**	0.052	0.049	0.052	0.048	0.047	0.054	**0.062**	**0.070**
KT	**0.063**	0.059	0.056	0.048	0.047	0.052	0.043	0.040	0.046	**0.062**	0.056
AS	**0.078**	**0.069**	**0.067**	0.051	0.047	0.051	0.043	0.041	0.051	**0.067**	**0.074**
SF	**0.075**	**0.071**	**0.065**	0.055	0.050	0.056	0.046	0.045	0.053	**0.069**	**0.074**
AP	**0.069**	**0.064**	0.059	0.048	0.047	0.049	0.044	0.039	0.050	**0.063**	**0.065**
RJ	**0.071**	**0.066**	0.060	0.051	0.047	0.051	0.043	0.041	0.050	**0.064**	**0.070**
T_1	**0.078**	**0.071**	**0.061**	0.052	0.048	0.053	0.047	0.047	0.052	**0.064**	**0.076**
JB	**0.074**	**0.065**	**0.061**	0.048	0.047	0.051	0.041	0.040	0.047	**0.068**	**0.068**
H1	**0.071**	**0.066**	**0.064**	0.052	0.051	0.054	0.047	0.049	0.054	**0.065**	**0.069**
CC	**0.077**	**0.068**	**0.063**	0.051	0.049	0.050	0.043	0.043	0.052	**0.068**	**0.075**
CS	**0.069**	**0.066**	0.059	0.051	0.050	0.052	0.049	0.048	0.054	0.060	**0.066**
AJB	**0.071**	**0.063**	0.060	0.049	0.047	0.050	0.041	0.040	0.047	**0.068**	**0.066**
ZA	**0.073**	**0.067**	**0.062**	0.052	0.049	0.053	0.044	0.046	0.052	**0.064**	**0.070**
ZC	**0.069**	**0.065**	**0.060**	0.050	0.049	0.051	0.047	0.044	0.052	**0.063**	**0.068**
β_3^2	0.056	0.054	0.053	0.050	0.047	0.052	0.046	0.046	0.052	0.058	0.053
H_n	0.046	0.043	0.046	0.049	0.048	0.051	0.058	**0.060**	**0.070**	**0.072**	**0.081**
X_{APD}	**0.068**	**0.062**	0.057	0.050	0.049	0.054	0.046	0.045	0.052	**0.062**	**0.064**
B_v	**0.065**	0.060	0.059	0.049	0.055	0.052	0.054	0.054	0.052	0.056	**0.061**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.088**	**0.074**	**0.067**	**0.062**	0.056	0.053	0.050	**0.063**	**0.069**	**0.075**	**0.086**
δ	0.047	0.052	0.048	0.047	0.048	0.050	0.048	0.056	**0.066**	**0.073**	**0.084**

Source: author's work.

Table 11. $\mathrm{LOG}(0.103, b)$ distribution. PoT versus γ_1 and $M(0.103, b; 1.106, \sigma)$ for $n = 25$

γ_1	0.290	0.350	0.400	0.450	0.500	0.550	0.600	0.650	0.700	0.750	0.800
$\bar{\gamma}_2$	0.149	0.219	0.286	0.362	0.448	0.543	0.647	0.761	0.884	1.017	1.159
b	0.096	0.116	0.132	0.148	0.164	0.180	0.196	0.211	0.226	0.242	0.256
σ	0.106	0.117	0.119	0.133	0.148	0.163	0.178	0.169	0.155	0.140	0.126
M	0.974	0.950	**0.900**	**0.900**	**0.900**	**0.900**	**0.900**	0.844	0.772	0.699	0.628
GoFT	PoT										
AD	**0.069**	**0.083**	**0.090**	**0.093**	**0.110**	**0.117**	**0.136**	**0.152**	**0.166**	**0.196**	**0.198**
SW	**0.082**	**0.088**	**0.102**	**0.108**	**0.131**	**0.138**	**0.164**	**0.183**	**0.199**	**0.236**	**0.238**
KT	**0.066**	**0.076**	**0.080**	**0.083**	**0.097**	**0.104**	**0.114**	**0.125**	**0.132**	**0.148**	**0.155**
AS	**0.084**	**0.092**	**0.107**	**0.121**	**0.141**	**0.155**	**0.175**	**0.197**	**0.212**	**0.251**	**0.255**
SF	**0.083**	**0.092**	**0.102**	**0.112**	**0.133**	**0.144**	**0.164**	**0.183**	**0.199**	**0.235**	**0.238**
AP	**0.078**	**0.085**	**0.101**	**0.105**	**0.123**	**0.135**	**0.151**	**0.168**	**0.178**	**0.209**	**0.211**
RJ	**0.080**	**0.088**	**0.099**	**0.107**	**0.128**	**0.138**	**0.158**	**0.176**	**0.192**	**0.229**	**0.231**
T_1	**0.082**	**0.092**	**0.108**	**0.119**	**0.138**	**0.153**	***0.180***	***0.203***	***0.223***	***0.261***	***0.265***
JB	**0.079**	**0.085**	**0.102**	**0.109**	**0.127**	**0.138**	**0.155**	**0.173**	**0.186**	**0.220**	**0.222**
H1	**0.075**	**0.085**	**0.091**	**0.096**	**0.117**	**0.126**	**0.145**	**0.160**	**0.171**	**0.208**	**0.211**
CC	**0.083**	**0.092**	**0.106**	**0.121**	**0.140**	**0.155**	**0.177**	**0.202**	**0.217**	**0.256**	**0.259**
CS	**0.079**	**0.087**	**0.099**	**0.106**	**0.128**	**0.135**	**0.161**	**0.181**	**0.196**	**0.234**	**0.234**
AJB	**0.078**	**0.082**	**0.100**	**0.103**	**0.120**	**0.131**	**0.147**	**0.163**	**0.174**	**0.203**	**0.206**
ZA	**0.079**	**0.087**	**0.101**	**0.112**	**0.132**	**0.143**	**0.166**	**0.187**	**0.203**	**0.243**	**0.248**
ZC	**0.080**	**0.087**	**0.102**	**0.109**	**0.130**	**0.141**	**0.162**	**0.183**	**0.200**	**0.235**	**0.239**
β_3^2	0.056	0.060	**0.061**	0.058	**0.062**	**0.067**	**0.071**	**0.071**	**0.069**	**0.074**	**0.076**
H_n	**0.088**	**0.102**	**0.112**	**0.114**	**0.137**	**0.139**	**0.165**	**0.186**	**0.201**	**0.234**	**0.234**
X_{APD}	**0.076**	**0.081**	**0.092**	**0.095**	**0.115**	**0.126**	**0.142**	**0.157**	**0.171**	**0.206**	**0.207**
B_v	**0.066**	**0.076**	**0.081**	**0.084**	**0.098**	**0.108**	**0.122**	**0.136**	**0.151**	**0.175**	**0.179**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.092**	**0.100**	**0.112**	**0.113**	**0.131**	**0.135**	**0.156**	**0.173**	**0.180**	**0.204**	**0.214**
δ	**0.086**	**0.104**	**0.115**	**0.116**	**0.138**	**0.146**	**0.176**	**0.193**	**0.205**	**0.241**	**0.244**

Source: author's work.

Table 12. $\mathrm{NM}_1(a,b,\omega)$ distribution. PoT versus γ_1 and $M(a,b,\omega;0{,}1)$ for $n=25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.522	0.522	0.449	0.471	0.110	0	0.110	0.471	0.449	0.522	0.522
a	–0.739	–0.524	–0.371	–0.216	–0.194	0	0.194	0.216	0.371	0.524	0.739
b	2.033	1.827	1.620	1.516	1.211	1	1.211	1.516	1.620	1.827	2.033
ω	0.317	0.352	0.395	0.516	0.551	1	0.551	0.516	0.395	0.352	0.317
M	0.750	0.800	0.850	0.900	0.950	1	0.950	0.900	0.850	0.800	0.750
GoFT	PoT										
AD	**0.113**	**0.099**	**0.080**	**0.069**	0.057	0.050	0.055	**0.075**	**0.080**	**0.097**	**0.113**
SW	**0.114**	**0.099**	**0.086**	**0.082**	**0.061**	0.052	0.058	**0.084**	**0.083**	**0.099**	**0.112**
KT	**0.106**	**0.102**	**0.097**	**0.096**	**0.061**	0.052	0.060	**0.101**	**0.092**	**0.098**	**0.111**
AS	**0.120**	**0.108**	**0.098**	**0.095**	**0.063**	0.052	**0.064**	**0.097**	**0.093**	**0.107**	**0.119**
SF	**0.131**	**0.122**	**0.100**	**0.097**	**0.065**	0.055	**0.066**	**0.102**	**0.100**	**0.118**	**0.135**
AP	**0.111**	**0.106**	**0.096**	**0.095**	**0.064**	0.054	**0.062**	**0.097**	**0.095**	**0.102**	**0.113**
RJ	**0.124**	**0.116**	**0.096**	**0.091**	**0.061**	0.051	**0.062**	**0.098**	**0.095**	**0.112**	**0.127**
T_1	**0.108**	**0.096**	**0.081**	**0.076**	0.059	0.051	**0.061**	**0.083**	**0.079**	**0.094**	**0.106**
JB	**0.121**	**0.117**	**0.104**	**0.103**	**0.066**	0.053	**0.063**	**0.107**	**0.103**	**0.114**	**0.127**
H1	**0.124**	**0.116**	**0.095**	**0.087**	**0.063**	0.053	**0.064**	**0.093**	**0.090**	**0.111**	**0.125**
CC	**0.119**	**0.109**	**0.097**	**0.091**	**0.061**	0.052	**0.064**	**0.096**	**0.093**	**0.110**	**0.122**
CS	**0.109**	**0.094**	**0.082**	**0.079**	0.060	0.053	0.058	**0.081**	**0.080**	**0.093**	**0.108**
AJB	<u>**0.121**</u>	<u>**0.118**</u>	<u>**0.107**</u>	<u>**0.103**</u>	<u>**0.067**</u>	0.054	<u>**0.062**</u>	<u>**0.110**</u>	<u>**0.103**</u>	<u>**0.111**</u>	<u>**0.129**</u>
ZA	**0.109**	**0.097**	**0.089**	**0.086**	**0.061**	0.053	0.060	**0.086**	**0.084**	**0.098**	**0.110**
ZC	**0.106**	**0.096**	**0.084**	**0.084**	**0.061**	0.053	0.058	**0.085**	**0.084**	**0.095**	**0.108**
β_3^2	**0.098**	**0.101**	**0.091**	**0.086**	0.058	0.052	0.059	**0.088**	**0.080**	**0.097**	**0.103**
H_n	**0.074**	**0.064**	0.057	0.053	0.053	0.053	0.055	**0.070**	**0.080**	**0.102**	**0.126**
X_{APD}	**0.119**	**0.108**	**0.093**	**0.087**	**0.062**	0.053	0.059	**0.095**	**0.089**	**0.101**	**0.119**
B_v	**0.078**	**0.074**	**0.065**	0.058	0.054	0.049	0.054	0.060	**0.065**	**0.076**	**0.081**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.128**	**0.107**	**0.085**	**0.068**	0.059	0.049	0.058	**0.074**	**0.083**	**0.104**	**0.128**
δ	**0.084**	**0.074**	**0.065**	0.060	0.054	0.051	0.059	**0.080**	**0.087**	**0.113**	**0.130**

Source: author's work.

Table 13. $\mathrm{NM}_2(a,b,\omega)$ distribution. PoT versus γ_1 and $M(a,b,\omega;0{,}1)=0.95$ for $n=25$

γ_1	−0.250	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.546	0.425	0.350	0.134	0.130	0	0.130	0.134	0.350	0.425	0.546
a	−0.621	−0.522	−0.396	−0.398	−0.180	0	0.180	0.398	0.396	0.522	0.621
b	1.474	1.418	1.384	1.219	1.230	1	1.230	1.219	1.384	1.418	1.474
ω	0.805	0.779	0.743	0.693	0.564	1	0.564	0.693	0.743	0.779	0.805
M	0.950	0.950	0.950	0.950	0.950	1	0.950	0.950	0.950	0.950	0.950
GoFT	PoT										
AD	**0.081**	**0.074**	**0.079**	0.059	0.057	0.050	0.056	**0.061**	**0.067**	**0.074**	**0.080**
SW	**0.097**	**0.088**	**0.088**	**0.067**	0.059	0.051	0.057	**0.063**	**0.075**	**0.081**	**0.094**
KT	**0.105**	**0.091**	**0.091**	**0.068**	0.058	0.051	0.058	**0.067**	**0.083**	**0.092**	**0.103**
AS	**0.109**	**0.100**	**0.097**	**0.070**	**0.064**	0.051	0.060	**0.068**	**0.085**	**0.094**	**0.104**
SF	**0.110**	**0.099**	**0.099**	**0.073**	**0.063**	0.053	**0.063**	**0.069**	**0.087**	**0.094**	**0.107**
AP	**0.108**	**0.097**	**0.095**	**0.068**	**0.063**	0.051	0.057	**0.067**	**0.084**	**0.094**	**0.107**
RJ	**0.106**	**0.094**	**0.095**	**0.070**	0.060	0.049	0.060	**0.065**	**0.082**	**0.089**	**0.102**
T_1	**0.092**	**0.084**	**0.084**	**0.064**	**0.062**	0.047	0.057	**0.063**	**0.073**	**0.079**	**0.090**
JB	<u>**0.115**</u>	<u>**0.102**</u>	<u>**0.100**</u>	<u>**0.071**</u>	<u>**0.064**</u>	0.052	**0.061**	**0.069**	**0.089**	**0.101**	**0.112**
H1	**0.095**	**0.084**	**0.088**	**0.066**	**0.062**	0.053	**0.061**	**0.064**	**0.074**	**0.086**	**0.094**
CC	**0.108**	**0.099**	**0.097**	**0.070**	**0.062**	0.049	0.060	**0.067**	**0.082**	**0.092**	**0.103**
CS	**0.093**	**0.085**	**0.084**	**0.064**	0.058	0.051	0.055	**0.062**	**0.074**	**0.078**	**0.092**
AJB	<u>**0.115**</u>	<u>**0.101**</u>	<u>**0.100**</u>	<u>**0.072**</u>	<u>**0.063**</u>	0.052	<u>**0.061**</u>	<u>**0.070**</u>	<u>**0.090**</u>	<u>**0.101**</u>	<u>**0.112**</u>
ZA	**0.100**	**0.090**	**0.086**	**0.068**	0.059	0.051	0.059	**0.067**	**0.079**	**0.084**	**0.100**
ZC	**0.100**	**0.093**	**0.089**	**0.068**	**0.060**	0.051	0.057	**0.064**	**0.079**	**0.083**	**0.098**
β_3^2	**0.084**	**0.078**	**0.078**	**0.062**	0.057	0.054	0.057	**0.061**	**0.072**	**0.080**	**0.084**
H_n	**0.082**	**0.079**	**0.085**	**0.066**	**0.064**	0.052	0.058	**0.066**	**0.069**	**0.078**	**0.083**
X_{APD}	**0.099**	**0.088**	**0.091**	**0.067**	0.059	0.048	0.058	**0.063**	**0.074**	**0.089**	**0.097**
B_v	**0.068**	**0.067**	**0.066**	0.058	0.056	0.051	0.052	0.056	**0.060**	**0.064**	**0.069**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	0.044	0.041	0.041	0.032	0.029	0.050	0.056	**0.066**	**0.070**	**0.079**	**0.085**
δ	**0.091**	**0.085**	**0.092**	**0.069**	**0.068**	0.050	0.060	**0.066**	**0.072**	**0.087**	**0.089**

Source: author's work.

Table 14. $SB(a,b)$ distribution. PoT versus γ_1 and $M(a,b;0{,}0.093)$ for $n=25$

γ_1	−0.250	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$						−0.038					
a	−0.810	−0.724	−0.599	−0.463	−0.262	0	0.262	0.463	0.599	0.724	0.810
b	1.913	2.055	2.187	2.396	2.576	2.669	2.576	2.396	2.187	2.055	1.913
M	0.543	0.540	0.534	0.526	0.514	0.500	0.486	0.474	0.466	0.460	0.457
GoFT						PoT					
AD	**0.061**	0.051	0.049	0.047	0.045	0.043	0.044	0.044	0.047	0.055	**0.061**
SW	**0.060**	0.048	0.047	0.043	0.039	0.038	0.041	0.043	0.046	0.053	0.058
KT	0.038	0.036	0.035	0.033	0.034	0.032	0.033	0.032	0.037	0.037	0.036
AS	0.045	0.038	0.037	0.035	0.030	0.030	0.030	0.034	0.034	0.041	0.045
SF	0.048	0.041	0.038	0.037	0.034	0.035	0.035	0.036	0.038	0.045	0.047
AP	0.041	0.039	0.036	0.034	0.030	0.031	0.032	0.034	0.036	0.041	0.041
RJ	0.044	0.038	0.035	0.034	0.031	0.032	0.031	0.033	0.035	0.041	0.043
T_1	0.059	0.047	0.044	0.042	0.037	0.036	0.039	0.043	0.044	0.052	0.058
JB	0.035	0.032	0.031	0.029	0.026	0.029	0.027	0.028	0.030	0.033	0.035
H1	0.053	0.047	0.045	0.042	0.039	0.039	0.040	0.040	0.045	0.050	0.051
CC	0.044	0.038	0.036	0.035	0.031	0.031	0.031	0.034	0.036	0.043	0.047
CS	**0.061**	0.050	0.048	0.045	0.039	0.039	0.041	0.044	0.049	0.055	**0.061**
AJB	0.033	0.030	0.028	0.029	0.025	0.026	0.027	0.027	0.028	0.031	0.032
ZA	0.054	0.044	0.044	0.041	0.036	0.035	0.038	0.041	0.044	0.051	0.054
ZC	0.057	0.047	0.043	0.041	0.035	0.035	0.038	0.042	0.044	0.052	0.056
β_3^2	0.038	0.040	0.042	0.042	0.041	0.041	0.038	0.039	0.044	0.045	0.038
H_n	0.045	0.038	0.042	0.043	0.047	0.048	0.053	0.056	**0.065**	**0.079**	**0.086**
X_{APD}	0.052	0.045	0.041	0.039	0.037	0.034	0.039	0.040	0.042	0.049	0.050
B_v	**0.070**	**0.061**	0.060	0.056	0.051	0.052	0.053	0.054	0.060	**0.066**	**0.070**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.081**	**0.072**	**0.065**	0.058	0.052	0.045	0.053	0.056	**0.064**	**0.078**	**0.085**
δ	0.044	0.039	0.040	0.040	0.043	0.044	0.048	0.050	0.059	**0.069**	**0.073**

Source: author's work.

Table 15. $SN(a)$ distribution. PoT versus γ_1 and $M(a;\mu,\sigma)$ for $n=25$

γ_1	−0.250	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.138	0.102	0.070	0.041	0.016	0	0.016	0.041	0.070	0.102	0.138
a	−1.349	−1.199	−1.043	−0.871	−0.659	0	0.659	0.871	1.043	1.199	1.349
μ	−0.252	−0.231	−0.262	−0.299	−0.339	0	0.476	0.677	0.825	0.879	1
σ	1.012	1.018	0.968	0.905	0.967	1	0.954	0.955	0.771	0.720	0.500
M	0.800	0.810	0.850	0.900	0.950	1	0.965	0.910	0.863	0.840	0.692
GoFT	PoT										
AD	**0.065**	**0.063**	0.052	0.052	0.050	0.053	0.053	0.059	0.059	**0.062**	**0.066**
SW	**0.071**	**0.066**	0.058	0.054	0.049	0.053	0.054	0.058	0.060	**0.067**	**0.073**
KT	**0.062**	**0.061**	0.055	0.054	0.048	0.049	0.053	0.054	0.056	0.060	**0.065**
AS	**0.075**	**0.069**	**0.061**	0.055	0.048	0.049	0.054	0.058	0.060	**0.070**	**0.077**
SF	**0.075**	**0.069**	**0.062**	0.056	0.051	0.052	0.058	**0.061**	**0.063**	**0.071**	**0.076**
AP	**0.070**	**0.065**	0.057	0.053	0.048	0.049	0.052	0.053	0.057	**0.065**	**0.071**
RJ	**0.071**	**0.065**	0.057	0.051	0.048	0.049	0.054	0.056	0.060	**0.066**	**0.072**
T_1	**0.074**	**0.068**	0.058	0.055	0.048	0.049	0.056	0.057	0.059	**0.067**	**0.075**
JB	**0.072**	**0.065**	0.059	0.053	0.048	0.049	0.052	0.054	0.059	**0.066**	**0.074**
H1	**0.069**	**0.067**	0.054	0.055	0.049	0.052	0.057	0.059	**0.061**	**0.067**	**0.070**
CC	**0.074**	**0.067**	0.060	0.054	0.048	0.048	0.055	0.057	0.060	**0.070**	**0.077**
CS	**0.071**	**0.065**	0.057	0.052	0.048	0.051	0.052	0.058	0.060	**0.065**	**0.071**
AJB	**0.069**	**0.065**	0.059	0.053	0.050	0.047	0.054	0.054	0.059	**0.066**	**0.073**
ZA	**0.071**	**0.068**	0.058	0.054	0.049	0.050	0.052	0.056	0.058	**0.067**	**0.075**
ZC	**0.071**	**0.065**	0.057	0.054	0.049	0.050	0.053	0.055	0.058	**0.066**	**0.075**
β_3^2	0.053	0.058	0.049	0.055	0.052	0.052	0.053	0.053	0.055	0.053	0.053
H_n	0.047	0.049	0.045	0.048	0.048	0.053	0.058	<u>**0.065**</u>	<u>**0.068**</u>	<u>**0.074**</u>	<u>**0.079**</u>
X_{APD}	**0.069**	**0.065**	0.054	0.053	0.048	0.051	0.056	0.057	0.058	**0.065**	**0.067**
B_v	**0.063**	**0.062**	0.053	0.054	0.050	0.054	0.055	0.054	0.055	0.060	0.060
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	<u>**0.086**</u>	<u>**0.072**</u>	<u>**0.066**</u>	<u>**0.062**</u>	0.055	0.051	0.054	<u>**0.067**</u>	<u>**0.069**</u>	<u>**0.072**</u>	<u>**0.083**</u>
δ	0.050	0.051	0.045	0.046	0.048	0.051	0.055	**0.063**	**0.065**	**0.074**	**0.081**

Source: author's work.

Table 16. $SU(b,c,d)$ distribution. PoT versus γ_1 and $M(b,c,d;\mu,\sigma)$ for $n=25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	–2.470	–2.010	–1.340	–0.646	–0.119	0.085	–0.119	–0.646	–1.340	–2.010	–2.470
b	0.529	0.619	0.738	0.906	1.192	1.375	1.192	0.906	0.738	0.619	0.529
c	9.322	7.823	5.909	3.972	2.038	0	–2.038	–3.972	–5.909	–7.820	–9.320
d	9.256	9.936	10.381	10.721	11.041	11.129	11.041	10.721	10.381	9.936	9.256
μ	–0.570	–0.500	–0.412	–0.321	–0.207	0	0.207	0.321	0.412	0.497	0.569
σ	0.104	0.092	0.087	0.097	0.111	0.124	0.111	0.097	0.087	0.092	0.104
M	0.750	0.800	0.851	0.900	0.950	0.998	0.950	0.900	0.851	0.800	0.750
GoFT	PoT										
AD	**0.066**	**0.066**	0.050	0.054	0.051	0.051	0.052	0.056	0.055	**0.063**	**0.064**
SW	**0.076**	**0.068**	0.055	0.055	0.048	0.052	0.051	0.058	0.057	**0.064**	**0.069**
KT	**0.069**	**0.061**	0.058	0.056	0.052	0.053	0.052	0.057	0.055	**0.061**	0.059
AS	**0.077**	**0.067**	0.059	0.059	0.054	0.055	0.052	0.059	0.056	**0.067**	**0.075**
SF	**0.079**	**0.067**	**0.061**	0.059	0.051	0.057	0.052	**0.063**	0.059	**0.067**	**0.071**
AP	**0.074**	**0.067**	0.059	0.058	0.055	0.055	0.052	0.060	0.055	**0.063**	**0.070**
RJ	**0.075**	**0.063**	0.056	0.055	0.046	0.054	0.049	0.059	0.056	**0.063**	**0.068**
T_1	**0.076**	**0.065**	0.055	0.058	0.052	0.054	0.047	0.057	0.055	**0.065**	**0.075**
JB	**0.076**	**0.067**	0.059	0.060	0.054	0.057	0.053	0.060	0.056	**0.065**	**0.070**
H1	**0.069**	**0.066**	0.058	0.055	0.050	0.056	0.052	0.058	0.056	**0.064**	**0.069**
CC	**0.078**	**0.066**	0.058	0.057	0.052	0.055	0.051	0.060	0.055	**0.068**	**0.073**
CS	**0.076**	**0.068**	0.054	0.054	0.048	0.052	0.051	0.057	0.056	**0.063**	**0.069**
AJB	**0.074**	**0.066**	0.058	0.060	0.055	0.057	0.053	0.060	0.055	**0.064**	**0.069**
ZA	**0.078**	**0.069**	0.056	0.054	0.049	0.053	0.049	0.057	0.057	**0.063**	**0.071**
ZC	**0.076**	**0.068**	0.057	0.056	0.050	0.051	0.049	0.059	0.056	**0.063**	**0.070**
β_3^2	0.054	0.058	0.054	0.055	0.050	0.055	0.054	0.056	0.055	0.055	0.055
H_n	0.046	0.047	0.042	0.045	0.047	0.051	0.057	**0.061**	**0.063**	**0.073**	**0.078**
X_{APD}	**0.070**	**0.062**	0.055	0.055	0.048	0.053	0.051	0.060	0.054	**0.064**	**0.066**
B_v	**0.065**	0.060	0.054	0.053	0.049	0.052	0.050	0.055	0.055	0.059	**0.063**
$LF_{\bar{\alpha},\bar{\beta}}$	**0.086**	**0.076**	**0.066**	**0.061**	0.057	0.052	0.057	**0.062**	**0.065**	**0.073**	**0.080**
δ	0.051	0.045	0.045	0.049	0.049	0.052	0.058	**0.061**	**0.068**	**0.072**	**0.082**

Source: author's work.

Table 17. $W(1.851, b)$ distribution. PoT versus γ_1 and $M(1.851, b; 1.673, 0.532)$ for $n = 25$

γ_1	−0.250	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	−0.124	−0.173	−0.214	−0.245	−0.269	−0.283	−0.289	−0.287	−0.276	−0.258	−0.230
b	4.971	4.634	4.334	4.064	3.822	3.602	3.403	3.222	3.056	2.905	2.766
M	0.849	0.880	0.911	0.940	0.968	0.985	0.974	0.951	0.927	0.904	0.882
GoFT	PoT										
AD	**0.065**	0.058	0.053	0.049	0.044	0.041	0.044	0.043	0.047	0.052	0.058
SW	**0.064**	0.055	0.052	0.047	0.041	0.038	0.038	0.039	0.043	0.053	0.060
KT	0.044	0.040	0.038	0.031	0.033	0.029	0.034	0.034	0.032	0.039	0.043
AS	0.059	0.051	0.041	0.033	0.029	0.029	0.028	0.029	0.034	0.042	0.053
SF	**0.061**	0.051	0.045	0.038	0.034	0.032	0.032	0.034	0.036	0.044	0.051
AP	0.048	0.045	0.040	0.031	0.031	0.029	0.030	0.032	0.034	0.041	0.049
RJ	0.056	0.048	0.042	0.034	0.033	0.029	0.030	0.030	0.033	0.041	0.048
T_1	**0.067**	**0.060**	0.051	0.046	0.040	0.035	0.036	0.037	0.042	0.050	0.060
JB	0.049	0.044	0.035	0.027	0.027	0.025	0.023	0.026	0.028	0.035	0.046
H1	0.058	0.052	0.049	0.044	0.039	0.038	0.039	0.041	0.040	0.047	0.056
CC	0.057	0.050	0.041	0.034	0.029	0.029	0.027	0.030	0.035	0.043	0.054
CS	**0.065**	0.055	0.054	0.049	0.042	0.040	0.041	0.042	0.045	0.055	0.060
AJB	0.046	0.042	0.033	0.025	0.026	0.023	0.023	0.026	0.027	0.034	0.042
ZA	**0.063**	0.054	0.049	0.043	0.038	0.035	0.036	0.037	0.037	0.047	0.055
ZC	**0.060**	0.052	0.048	0.043	0.039	0.035	0.037	0.039	0.041	0.049	0.058
β_3^2	0.042	0.043	0.043	0.042	0.041	0.038	0.041	0.041	0.038	0.041	0.041
H_n	0.048	0.043	0.043	0.043	0.043	0.045	0.053	0.055	0.058	**0.073**	**0.078**
X_{APD}	0.056	0.050	0.046	0.042	0.038	0.036	0.039	0.039	0.040	0.047	0.053
B_v	**0.064**	0.059	0.058	**0.065**	0.054	0.050	0.053	0.053	0.054	**0.062**	**0.067**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.090**	**0.078**	**0.065**	**0.063**	0.052	0.039	0.049	0.052	0.058	**0.069**	**0.077**
δ	0.047	0.041	0.044	0.041	0.041	0.041	0.048	0.049	0.054	**0.063**	**0.072**

Source: author's work.

Table 18. $ES_1(\gamma_1, \bar{\gamma}_2)$ distribution. PoT versus γ_1 and $M(\gamma_1, \bar{\gamma}_2; 0{,}1)$ for $n = 25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.250	0.200	0.150	0.100	0.050	0	0.050	0.100	0.150	0.200	0.250
M	0.966	0.973	0.979	0.986	0.993	1	0.993	0.986	0.979	0.973	0.966
GoFT	PoT										
AD	**0.073**	**0.068**	**0.060**	0.056	0.050	0.052	0.052	0.057	0.060	**0.064**	**0.071**
SW	**0.081**	**0.072**	**0.062**	0.058	0.053	0.051	0.057	0.056	**0.063**	**0.070**	**0.082**
KT	**0.075**	**0.071**	**0.063**	0.057	0.057	0.049	0.051	0.060	**0.064**	**0.064**	**0.080**
AS	**0.087**	**0.080**	**0.064**	0.057	0.055	0.053	0.057	**0.062**	**0.066**	**0.079**	**0.085**
SF	**0.089**	**0.077**	**0.067**	**0.063**	0.058	0.053	0.060	**0.061**	**0.068**	**0.077**	**0.088**
AP	**0.083**	**0.076**	**0.065**	0.060	0.054	0.054	0.053	**0.061**	**0.065**	**0.073**	**0.083**
RJ	**0.083**	**0.072**	**0.063**	0.059	0.054	0.049	0.056	0.057	**0.064**	**0.073**	**0.082**
T_1	**0.080**	**0.074**	0.060	0.054	0.050	0.051	0.058	0.059	**0.062**	**0.073**	**0.076**
JB	**0.086**	**0.079**	**0.068**	0.059	0.056	0.054	0.055	**0.062**	**0.068**	**0.077**	**0.087**
H1	**0.080**	**0.074**	**0.062**	0.058	0.052	0.055	0.057	0.059	**0.065**	**0.071**	**0.081**
CC	**0.088**	**0.081**	**0.063**	0.057	0.054	0.053	0.058	**0.061**	**0.065**	**0.076**	**0.085**
CS	**0.078**	**0.071**	**0.061**	0.058	0.051	0.050	0.055	0.055	**0.060**	**0.067**	**0.079**
AJB	**0.085**	**0.080**	**0.067**	0.060	0.056	0.053	0.055	**0.063**	**0.067**	**0.075**	**0.089**
ZA	**0.080**	**0.073**	**0.061**	0.056	0.054	0.048	0.059	0.056	**0.063**	**0.072**	**0.082**
ZC	**0.078**	**0.073**	**0.060**	0.057	0.051	0.049	0.055	0.055	**0.065**	**0.072**	**0.081**
β_3^2	**0.063**	**0.062**	0.058	0.055	0.054	0.050	0.049	0.059	0.057	0.058	**0.069**
H_n	0.050	0.050	0.047	0.047	0.046	0.053	0.057	**0.060**	**0.065**	**0.073**	**0.083**
X_{APD}	**0.078**	**0.070**	**0.062**	0.059	0.051	0.050	0.054	0.055	**0.063**	**0.067**	**0.077**
B_v	**0.063**	**0.065**	0.057	0.054	0.051	0.055	0.051	0.053	0.055	0.060	**0.065**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.085**	**0.079**	**0.069**	0.058	0.055	0.051	0.058	**0.066**	**0.066**	**0.079**	**0.085**
δ	0.052	0.050	0.050	0.049	0.048	0.050	0.056	**0.061**	**0.067**	**0.078**	**0.088**

Source: author's work.

Table 19. $ES_1(\gamma_{1,},\bar{\gamma}_2)$ distribution. PoT versus γ_1 and $M(\gamma_{1,},\bar{\gamma}_2 = 0; 0{,}1)$ for $n = 25$

γ_1 $\bar{\gamma}_2$ M	−0.205 0.969	−0.200 0.975	−0.150 0.981	−0.100 0.987	−0.050 0.994	0 0 1	0.050 0.994	0.100 0.987	0.150 0.981	0.200 0.975	0.250 0.969
GoFT						PoT					
AD	**0.063**	0.051	0.054	0.050	0.047	0.050	0.049	0.052	0.052	0.058	**0.061**
SW	**0.065**	0.056	0.054	0.053	0.048	0.051	0.049	0.052	0.053	0.060	**0.064**
KT	0.051	0.049	0.048	0.051	0.046	0.051	0.048	0.047	0.050	0.051	0.053
AS	**0.063**	0.058	0.056	0.055	0.049	0.051	0.050	0.051	0.053	**0.062**	**0.065**
SF	**0.065**	0.055	0.057	0.056	0.049	0.053	0.051	0.052	0.055	0.060	**0.066**
AP	0.056	0.053	0.053	0.054	0.045	0.051	0.049	0.048	0.050	0.056	0.060
RJ	0.059	0.053	0.053	0.053	0.045	0.049	0.048	0.048	0.051	0.056	**0.061**
T_1	<u>**0.067**</u>	<u>0.058</u>	<u>0.056</u>	<u>0.054</u>	0.049	0.047	0.048	0.052	0.054	**0.062**	**0.069**
JB	0.057	0.053	0.054	0.053	0.046	0.052	0.049	0.049	0.050	0.057	0.059
H1	**0.065**	0.053	0.056	0.053	0.050	0.053	0.052	0.052	0.055	0.058	**0.064**
CC	**0.062**	0.057	0.055	0.054	0.047	0.049	0.050	0.052	0.053	**0.062**	**0.066**
CS	**0.064**	0.055	0.054	0.052	0.048	0.051	0.048	0.051	0.052	0.058	**0.062**
AJB	0.056	0.052	0.053	0.053	0.047	0.052	0.048	0.047	0.051	0.055	0.059
ZA	**0.065**	0.055	0.053	0.052	0.047	0.051	0.047	0.048	0.052	0.057	**0.061**
ZC	**0.064**	0.056	0.053	0.053	0.047	0.051	0.047	0.049	0.053	0.056	0.059
β_3^2	0.047	0.044	0.048	0.050	0.048	0.054	0.052	0.048	0.048	0.047	0.047
H_n	0.043	0.041	0.046	0.044	0.047	0.052	0.054	0.057	**0.061**	**0.070**	**0.076**
X_{APD}	0.060	0.052	0.054	0.052	0.045	0.048	0.048	0.047	0.051	0.054	0.059
B_v	0.060	0.052	0.053	0.051	0.050	0.051	0.049	0.051	0.056	0.056	0.059
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	0.033	0.036	0.041	0.040	0.046	0.050	0.055	<u>0.055</u>	<u>**0.061**</u>	<u>**0.075**</u>	<u>**0.075**</u>
δ	0.046	0.039	0.045	0.044	0.046	0.050	0.055	0.056	**0.062**	**0.068**	**0.077**

Source: author's work.

Table 20. $P_1(\gamma_{1,}, \bar{\gamma}_2)$ distribution. PoT versus γ_1 and $M(\gamma_{1,}, \bar{\gamma}_2 > 0; 0{,}1)$ for $n = 25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.250	0.200	0.150	0.100	0.050	0	0.050	0.100	0.150	0.200	0.250
M	0.969	0.975	0.981	0.987	0.993	1	0.993	0.987	0.981	0.975	0.969
GoFT	PoT										
AD	**0.070**	**0.065**	**0.061**	0.056	0.054	0.052	0.052	0.056	**0.061**	**0.068**	**0.071**
SW	**0.078**	**0.070**	**0.065**	0.056	0.054	0.051	0.057	0.058	**0.064**	**0.074**	**0.078**
KT	**0.073**	**0.067**	**0.063**	0.055	0.055	0.049	0.055	**0.062**	**0.065**	**0.069**	**0.079**
AS	**0.082**	**0.073**	**0.068**	0.058	0.055	0.053	0.055	**0.062**	**0.069**	**0.078**	**0.086**
SF	**0.086**	**0.076**	**0.069**	0.057	0.057	0.053	0.058	**0.063**	**0.068**	**0.077**	**0.087**
AP	**0.079**	**0.072**	**0.067**	0.056	0.055	0.054	0.056	**0.062**	**0.070**	**0.075**	**0.084**
RJ	**0.081**	**0.071**	**0.065**	0.055	0.053	0.049	0.054	0.059	**0.064**	**0.073**	**0.081**
T_1	**0.081**	**0.071**	**0.061**	0.059	0.055	0.051	0.055	0.058	**0.064**	**0.077**	**0.077**
JB	**0.080**	**0.074**	**0.067**	0.057	0.056	0.054	0.055	**0.064**	**0.071**	**0.076**	**0.088**
H1	**0.076**	**0.069**	**0.064**	0.057	0.054	0.055	0.057	0.056	**0.063**	**0.071**	**0.078**
CC	**0.081**	**0.073**	**0.067**	0.058	0.054	0.053	0.055	**0.062**	**0.067**	**0.078**	**0.083**
CS	**0.077**	**0.069**	**0.063**	0.055	0.053	0.050	0.057	0.057	**0.063**	**0.071**	**0.076**
AJB	**0.077**	**0.075**	**0.066**	0.058	0.055	0.053	0.055	**0.065**	**0.071**	**0.075**	**0.089**
ZA	**0.081**	**0.070**	**0.066**	0.056	0.054	0.048	0.057	0.058	**0.065**	**0.077**	**0.080**
ZC	**0.079**	**0.070**	**0.064**	0.055	0.053	0.049	0.059	**0.061**	**0.065**	**0.075**	**0.081**
β_3^2	**0.060**	0.060	0.058	0.054	0.053	0.050	0.054	0.056	0.055	**0.061**	**0.065**
H_n	0.049	0.050	0.049	0.047	0.052	0.053	0.055	**0.064**	**0.065**	**0.076**	**0.081**
X_{APD}	**0.074**	**0.069**	**0.062**	0.056	0.054	0.050	0.056	0.058	**0.063**	**0.072**	**0.077**
B_v	**0.067**	**0.063**	0.057	0.053	0.054	0.055	0.054	0.053	0.056	**0.066**	**0.064**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.081**	**0.076**	**0.071**	**0.065**	0.057	0.051	0.055	**0.060**	**0.070**	**0.077**	**0.084**
δ	0.053	0.050	0.051	0.049	0.051	0.050	0.054	**0.063**	**0.068**	**0.079**	**0.085**

Source: author's work.

Table 21. $P_2(\gamma_{1,}, \bar{\gamma}_2)$ distribution. PoT versus γ_1 and $M(\gamma_{1,}, \bar{\gamma}_2 = 0; 0{,}1)$ for $n = 25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$						0					
M	0.967	0.974	0.981	0.987	0.994	1	0.994	0.987	0.981	0.974	0.967
GoFT	PoT										
AD	0.058	**0.060**	0.054	0.049	0.050	0.052	0.049	0.051	0.052	0.056	**0.065**
SW	**0.062**	0.060	0.057	0.050	0.051	0.051	0.051	0.051	0.052	0.058	**0.068**
KT	0.051	0.051	0.053	0.048	0.053	0.049	0.048	0.050	0.046	0.053	0.055
AS	**0.062**	0.059	0.056	0.050	0.052	0.053	0.046	0.051	0.050	0.059	**0.067**
SF	**0.060**	**0.065**	0.058	0.052	0.055	0.053	0.052	0.052	0.053	0.057	**0.067**
AP	0.058	0.055	0.055	0.049	0.051	0.054	0.049	0.049	0.049	0.056	**0.063**
RJ	0.057	0.059	0.054	0.048	0.051	0.049	0.049	0.048	0.050	0.054	**0.063**
T_1	**0.064**	**0.062**	0.057	0.051	0.050	0.051	0.046	0.050	0.051	**0.061**	**0.069**
JB	0.057	0.055	0.054	0.048	0.051	0.054	0.047	0.052	0.048	0.056	**0.062**
H1	0.058	**0.060**	0.058	0.050	0.055	0.055	0.048	0.054	0.051	0.056	**0.065**
CC	**0.061**	0.060	0.054	0.049	0.052	0.053	0.046	0.052	0.050	0.060	**0.068**
CS	**0.064**	**0.061**	0.056	0.049	0.051	0.050	0.050	0.050	0.052	0.058	**0.068**
AJB	0.053	0.054	0.053	0.046	0.052	0.053	0.049	0.051	0.046	0.055	0.059
ZA	**0.061**	0.058	0.054	0.050	0.049	0.048	0.049	0.049	0.048	0.059	**0.067**
ZC	0.059	0.060	0.057	0.050	0.051	0.049	0.050	0.049	0.050	0.060	**0.066**
β_3^2	0.046	0.048	0.051	0.048	0.052	0.050	0.051	0.052	0.048	0.047	0.048
H_n	0.042	0.048	0.044	0.047	0.047	0.053	0.051	0.058	**0.061**	**0.072**	**0.080**
X_{APD}	0.057	0.057	0.056	0.048	0.052	0.050	0.051	0.051	0.050	0.056	**0.062**
B_v	0.058	0.057	0.055	0.051	0.051	0.055	0.051	0.053	0.051	0.060	**0.064**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.079**	**0.077**	**0.068**	0.059	0.053	0.051	0.049	0.058	**0.062**	**0.074**	**0.082**
δ	0.041	0.048	0.044	0.044	0.047	0.050	0.049	0.056	0.060	**0.070**	**0.081**

Source: author's work.

Table 22. $P_3(\gamma_{1,}, \bar{\gamma}_2)$ distribution. PoT versus γ_1 and $M(\gamma_{1,}, \bar{\gamma}_2 < 0; 0{,}1)$ for $n = 25$

γ_1	−0.250	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	−0.250	−0.200	−0.150	−0.100	−0.050	0	−0.050	−0.100	−0.150	−0.200	−0.250
M	0.957	0.967	0.977	0.985	0.993	1	0.993	0.985	0.977	0.967	0.957
GoFT	PoT										
AD	0.057	0.050	0.051	0.051	0.049	0.052	0.047	0.052	0.050	0.051	0.059
SW	0.056	0.049	0.052	0.050	0.048	0.051	0.048	0.051	0.048	0.050	**0.061**
KT	0.036	0.039	0.042	0.046	0.041	0.049	0.045	0.043	0.041	0.038	0.044
AS	0.044	0.043	0.048	0.046	0.046	0.053	0.045	0.043	0.041	0.042	0.053
SF	0.048	0.044	0.047	0.049	0.049	0.053	0.048	0.047	0.044	0.045	0.053
AP	0.040	0.042	0.044	0.042	0.045	0.054	0.045	0.043	0.040	0.041	0.050
RJ	0.045	0.040	0.044	0.045	0.045	0.049	0.044	0.044	0.040	0.042	0.050
T_1	0.058	0.049	0.051	0.050	0.050	0.051	0.048	0.048	0.046	0.049	0.059
JB	0.035	0.036	0.043	0.042	0.045	0.054	0.045	0.041	0.038	0.036	0.045
H1	0.051	0.047	0.049	0.050	0.049	0.055	0.047	0.052	0.045	0.048	0.057
CC	0.044	0.040	0.048	0.045	0.046	0.053	0.045	0.044	0.042	0.045	0.054
CS	0.057	0.051	0.053	0.051	0.048	0.050	0.048	0.052	0.048	0.052	**0.061**
AJB	0.032	0.036	0.041	0.042	0.044	0.053	0.043	0.041	0.037	0.035	0.041
ZA	0.056	0.047	0.049	0.047	0.048	0.048	0.047	0.050	0.047	0.047	0.058
ZC	0.052	0.049	0.050	0.048	0.047	0.049	0.045	0.048	0.046	0.049	0.060
β_3^2	0.042	0.042	0.042	0.048	0.047	0.050	0.047	0.048	0.043	0.041	0.046
H_n	0.043	0.041	0.042	0.048	0.044	0.053	0.055	0.059	**0.063**	**0.070**	**0.079**
X_{APD}	0.050	0.044	0.046	0.047	0.046	0.050	0.044	0.046	0.043	0.047	0.053
B_v	<u>**0.068**</u>	<u>0.059</u>	0.052	0.054	0.053	0.055	0.052	0.055	0.053	0.059	**0.069**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	0.033	0.033	0.036	0.041	0.042	0.051	0.053	<u>**0.061**</u>	<u>**0.063**</u>	<u>**0.068**</u>	<u>**0.081**</u>
δ	0.044	0.037	0.044	0.043	0.044	0.050	0.052	0.058	0.060	**0.062**	**0.075**

Source: author's work.

Tables 6–22 show that when alternatives are asymmetric with non-constant γ_1, the GoFT for normality detects positive or negative γ_1 differently, depending on the alternative. For distribution B, three and six analysed GoFTs detect $\gamma_1 \leq -0.25$ and $\gamma_1 \geq 0.25$, respectively. For χ^2, all the analysed GoFTs, except β_3^2, detect $\gamma_1 \geq 0.294$. For G, all the analysed GoFTs, except β_3^2, detect $\gamma_1 \geq 0.223$. For LOG, all analysed GoFTs, except β_3^2, detect $\gamma_1 \geq 0.29$. For GP, the $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$ and $\mathrm{H_n}$ tests detect $\gamma_1 \leq -0.1$ and $\gamma_1 \geq 0$, respectively. For LCN, the $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$ test detects $\gamma_1 \leq -0.1$ or $\gamma_1 \geq 0.1$. For NM (see Table 12), thirteen and nine GoFTs detect $\gamma_1 \leq -0.1$ and $\gamma_1 \geq 0.1$, respectively. For NM with $M = 0.95$ (see Table 13), eleven and three GoFTs detect $\gamma_1 \leq -0.1$ and $\gamma_1 \geq 0.1$, respectively. For SB, $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$ and $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}, H_n$, the GoFTs detect $\gamma_1 \leq -0.15$ and $\gamma_1 \geq 0.15$, respectively. For SN and SU, the $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$ GoFT detects $|\gamma_1| \geq 0.1$. For W, the $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}, B_V$ tests detect $\gamma_1 \leq -0.1$ and $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}, B_V$ detects $\gamma_1 \geq 0.2$. For ES (see Table 18), only the S_F tests detect $\gamma_1 \leq -0.1$ and most tests detect $\gamma_1 \geq 0.15$. For ES with $\bar{\gamma}_2 = 0$ (see Table 19), only 10 tests detect $\gamma_1 \leq$

-0.25 and the $H_n, \mathrm{LF}_{\overline{\alpha},\overline{\beta}}, \delta$ tests detect $\gamma_1 \geq 0.15$. For P (see Table 20), the $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ GoFT detects $|\gamma_1| \geq 0.1$. For P (see Table 21), the $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ GoFT detects $|\gamma_1| \geq 0.15$. For P (see Table 22), only the B_v test detects $\gamma_1 \leq -0.25$ and the $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ test detects $\gamma_1 \geq 0.1$. As shown in Table 23, the H_n test best detects $\gamma_1 > 0$ for seven alternatives; the $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ test best detects $\gamma_1 < 0$ and $\gamma_1 > 0$ for nine and eight alternatives, respectively. The JB, AJB tests best detect $\gamma_1 \neq 0$ for two alternatives. The $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ and H_n tests best detect $\gamma_1 > 0$ for 13 alternative cases out of 17 (except LOG, NM_1, NM_2 and P_1). The $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ test best detects $\gamma_1 > 0$ for B, GP, LCN, SB, SN, SU, W, P_1, P_2. The JB, AJB tests best detect $\gamma_1 \neq 0$ for NM_1, NM_2 and $\gamma_1 > 0$ for the P_1 alternative. See Table 23 for more details.

Table 23. Summary of the results from Tables 6–22 for the analysed alternatives (A). The symbol in bold denotes $\bar{\gamma}_2 > 0$.

A	$\gamma_1 < 0$	$\gamma_1 > 0$	**A**	$\gamma_1 < 0$	$\gamma_1 > 0$
B	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	H_n	SN	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$
χ^2	n/a	H_n	SU	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}, \delta$
	n/a	H_n	W	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	H_n
GP	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	H_n	ES_1	SF	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$
LCN	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}, H_n$	ES_2	T_{1n}	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$
LOG	n/a	T_{1n}	P_1	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}, SF$	SF, JB, AJB
NM_1	SF, JB, AJB	SF, JB, AJB	P_2	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$
NM_2	JB, AJB	JB, AJB	P_3	B_v	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$
SB	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}, H_n$			

Source: author's work.

5. Summary and conclusions

The article contributes to the expansion of knowledge on GoFTs for normality. The study considers situations where the alternatives are asymmetric with non-constant skewness. At first, GoFTs were assessed with respect to their ability to detect samples for two reasons:

- they come from general populations where the alternatives with skewness values are close to zero or where the lowest possible skewness values occur, and
- the value of the normal-alternative similarity measure is close to unity.

Having already assessed the abilities of GoFTs, 21 of them were selected as a set of GoFTs to be applied to detect asymmetric alternatives with non-constant skewness.

Subsequently, a set of 13 alternatives was formed. These were distinguished as useful in deviation-from-normality-oriented Monte Carlo studies. Among them were alternatives of only negative skewness, only positive skewness or both negative and positive skewness. The alternatives in question fall into two categories: monolithic and compound distributions.

When describing a given distribution, the main emphasis was placed on defining formulas for skewness and its range. The (global) values of the similarity measure of the alternative to the normal distribution were determined.

The Monte Carlo study revealed that when alternatives are asymmetric with non-constant γ_1, GoFTs for normality detect positive or negative γ_1 differently, depending on the alternative. The H_n test best detects $\gamma_1 > 0$ for seven alternatives; the $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ test best detects $\gamma_1 < 0$ and $\gamma_1 > 0$ for nine and eight alternatives, respectively. The JB, AJB tests best detect $\gamma_1 \neq 0$ for two alternatives.

The $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ and H_n tests best detect $\gamma_1 > 0$ in 13 alternative cases out of 17 (except the LOG, NM_1, NM_2 and P_1 alternatives). The $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ test best detects $\gamma_1 > 0$ for $B, GP, LCN, SB, SN, SU, W, P_1, P_2$. The JB, AJB tests best detect $\gamma_1 \neq 0$ for NM_1, NM_2 and $\gamma_1 > 0$ for alternative P_1.

The $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ and H_n GoFTs best detect asymmetric distributions that deviate from normality due to small skewness, equal to even 0.05.

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Appendix

ES distribution

The Edgeworth series (ES) distribution is defined as:

$$f_{ES}(x) = \phi(x;0,1)\left[1+\sum_{i=3}^{\infty}\frac{1}{i!}\chi_i H_i(x)\right], \tag{A1}$$

where χ_i $(i = 3,4,\dots)$ are cumulants and $H_i(x)$ $(i = 3,4,\dots)$ are the probabilist's Hermite polynomials defined by recurrence relations

$$H_0(x) = 1, H_1(x) = x, H_2(x) = x^2 - 1, \dots, H_{n+1}(x) = xH_n(x) - nH_{n-1}(x).$$

For the purposes of the simulation, we need the first three terms of the series. Then (A1) takes the following form:

$$f_{ES}(x) = \phi(x;0,1)\left(1+\frac{1}{3!}\chi_3 H_3(x)+\frac{1}{4!}\chi_4 H_4(x)\right), \tag{A2}$$

where $H_3(x) = x^3 - 3x, H_4(x) = x^4 - 6x^2 + 3$ and $\chi_3 = \gamma_{1,}\, \chi_4 = \bar{\gamma}_2$. The PDF of the ES distribution based on (A2), is given by:

$$f_{ES}(x;\gamma_{1,},\bar{\gamma}_2) = \phi(x;0,1)\left(1+\tfrac{1}{3!}\gamma_{1,}(x^3-3x)+\tfrac{1}{4!}\bar{\gamma}_2(x^4-6x^2+3)\right).$$

P distribution

The P distribution is defined as:

$$f_P(x;\gamma_{1,},\bar{\gamma}_2) = exp\left[-\int\frac{x+b}{ax^2+bx+c}dx\right], \tag{A3}$$

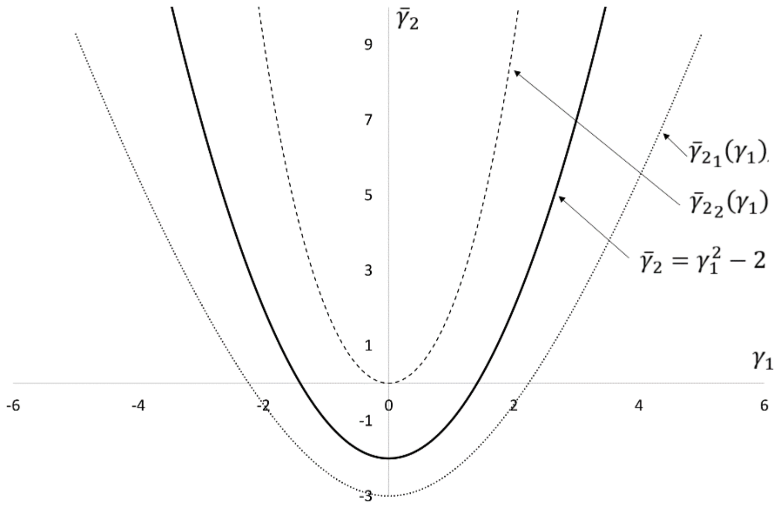
where a, b, c are given by (2). Let us consider three cases determined by the sign of the discriminant (and hence the number of real roots) of $ax^2 + bx + c$.

Case 1. $\Delta = 0 \Leftrightarrow b^2 = 4ac$. When solving this equation, we obtain the following:

$$\bar{\gamma}_{2_1} = \frac{6\sqrt{\left(\gamma_1^2+4\right)^3}-42\gamma_1^2+48}{\gamma_1^2-32},\quad \bar{\gamma}_{2_2} = \frac{-6\sqrt{\left(\gamma_1^2+4\right)^3}-42\gamma_1^2+48}{\gamma_1^2-32}.$$

The figure shows that the graph of function $\bar{\gamma}_{2_1}(\gamma_1)$ is located outside Malakhov's area $\bar{\gamma}_2 \geq \gamma_1^2 - 2$ (Malachov, 1978).

Figure. Excess kurtosis as a function of skewness when $b^2 = 4ac$



Source: author's work.

When substituting $\bar{\gamma}_2$ to (2), we obtain the following:

$$a = \frac{\gamma_1^4 - 4\gamma_1^2 + 4\sqrt{(\gamma_1^2+4)^3} - 32}{4\left[\gamma_1^4 + 2\gamma_1^2 + 5\sqrt{(\gamma_1^2+4)^3} - 8\right]}, \quad b = \frac{|\gamma_1|\left[6\gamma_1^2 + \sqrt{(\gamma_1^2+4)^3} + 24\right]}{2\left[\gamma_1^4 + 2\gamma_1^2 + 5\sqrt{(\gamma_1^2+4)^3} - 8\right]}, \quad c = \frac{\gamma_1^4 + 20\gamma_1^2 + 8\sqrt{(\gamma_1^2+4)^3} + 64}{4\left[\gamma_1^4 + 2\gamma_1^2 + 5\sqrt{(\gamma_1^2+4)^3} - 8\right]}.$$

The integral in (A3) can be written as:

$$\int \frac{x+b}{ax^2+bx+c} dx = 2\int \frac{dx}{2ax+b} + 2b\int \frac{2a-1}{(2ax+b)^2} dx,$$

then

$$\int \frac{x+b}{ax^2+bx+c} dx = \frac{ln(2ax+b)}{a} + \frac{b-2ab}{a(2ax+b)} + C_1.$$

The PDF of the P distribution based on (A3) is given by:

$$f_P\left(x;\gamma_{1,},\bar{\gamma}_2\right) = \frac{1}{C_2(2ax+b)^{1/a}} exp\left[\frac{2ab-b}{a(2ax+b)}\right],$$

where C_2 is given by (3).

Case 2. $\Delta < 0 \Leftrightarrow b^2 < 4ac$.

The integral in (A3) can be written as:

$$\int \frac{x+b}{ax^2+bx+c}dx = \frac{1}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx + \frac{2ab-b}{2a}\int \frac{dx}{ax^2+bx+c},$$

then

$$\int \frac{x+b}{ax^2+bx+c}dx = \frac{ln(ax^2+bx+c)}{2a} + \frac{2ab-b}{a\sqrt{4ac-b^2}} tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) + C_3.$$

The PDF of the P distribution based on (A3) is given by:

$$f_P\left(x;\gamma_{1,},\bar{\gamma}_2\right) = \frac{exp\left[\frac{b-2ab}{a\sqrt{4ac-b^2}} tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right]}{C_4(ax^2+bx+c)^{1/(2a)}},$$

where C_4 is given by (4).

Case 3. $\Delta > 0 \Leftrightarrow b^2 > 4ac$.

The integral in (A3) can be written as

$$\int \frac{x+b}{ax^2+bx+c}dx = \frac{1}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx + \frac{2ab-b}{2a}\int \frac{dx}{ax^2+bx+c} = I_1 + I_2, \tag{A4}$$

where

$$I_1 = \frac{1}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx = \frac{ln(ax^2+bx+c)}{2a} + C_5,$$

$$I_2 = \frac{2ab-b}{2a}\int \frac{dx}{ax^2+bx+c}.$$

Since

$$\frac{1}{ax^2+bx+c}=\frac{2a}{\sqrt{\Delta}}\left(\frac{1}{2ax+b-\sqrt{\Delta}}-\frac{1}{2ax+b+\sqrt{\Delta}}\right),$$

then

$$I_2=\frac{2ab-b}{2a\sqrt{\Delta}}ln\left(\frac{2ax+b-\sqrt{\Delta}}{2ax+b+\sqrt{\Delta}}\right)+C_6.$$

The integral in (A3), based on (A4), is given by:

$$\int\frac{x+b}{ax^2+bx+c}dx=\frac{ln(ax^2+bx+c)}{2a}+\frac{2ab-b}{2a\sqrt{\Delta}}ln\left(\frac{2ax+b-\sqrt{\Delta}}{2ax+b+\sqrt{\Delta}}\right)+C_7.$$

The PDF of the P distribution based on (A3) is given by:

$$f_P\left(x;\gamma_{1,},\bar{\gamma}_2\right)=\frac{\left(\frac{2ax+b-\sqrt{\Delta}}{2ax+b+\sqrt{\Delta}}\right)^{\frac{b-2ab}{2a\sqrt{\Delta}}}}{C_8(ax^2+bx+c)^{1/(2a)}},$$

where C_8 is given by (5).

R codes

```
h=function(x) ((x-1)/(x+1))^2
Hn=function(x) {
   x=sort((x-mean(x))/sd(x))
   n=length(x)
   Fn=1+1:n/n
   F1=pnorm(x,0,1)+1
   return(mean(h(F1/Fn))) }

Fn=function(i,n,a,b) ((i - a)/(n - a - b + 1))
LF=function(x,alfa,beta) {
   x=sort(x); n=length(x)
   F=pnorm(x, mean(x), sd(x))
   return(max(abs(Fn(1:n,n,alfa,beta)-F))) }

RJ=function(x) {}
   x=sort(x); n=length(x)
   z=qnorm(Fn1(1:n,n,3/8,3/8),0,1); s1=sum(x*z); s2=sum(z*z)
   return(s1/sqrt(s2*(n-1)*var(x))) }
```

```
W1=function(u) qnorm(u)^2-1
T1n=function(x) {
  x=sort(x); n=length(x)
  if (n==25) A1=-0.2114 else A1=-0.1297
  if (n==25) B1=0.2323 else B1=0.34
  s=sd(x)*sqrt((n-1)/n)
  Fn=1:n/(n+1)
  Cn=sum((W1(Fn)-A1)*x)/sqrt(n)
  return(Cn^2/s^2/B1) }

TestSigma=function(x) {
  x=sort(x); Ft=pnorm(x,mean(x),sd(x))
  n=length(x); Fn=1:n/n
  licz=sum((abs(Ft-Fn))); mian=0
  for (i in 1:n) {
    mian=mian+max(Ft[i],Fn[i]) }
  return(licz/mian) }

Bv=function(x) {
  x=sort(x); n=length(x)
  mi=mean(x); sdev=sd(x)*sqrt((n-1)/n)
  if (n==25) m=5 else m=15; s=0
  for (i in 1:n) {
    up=i-m; if (up<1) up=1
    uk=i+m; if(uk>n) uk=n
    a=2*m/(x[uk]-x[up])/n
    b=exp(-0.5*((x[i]-mi)/sdev)^2)/sdev/sqrt(2*pi)
    s=s+((a-b)/(a+b))^2 }
  return(s/n) }
rGP=function(n,a,b) {
  x=numeric(n)
  for (i in 1:n){
    W=rgamma(1,1/b)
    d=dG(a,b)
    V=(W/d)^(1/b)
    x[i]=ifelse(runif(1,0,1)<1-a,(1-a)*V,-a*V) }
  return(sort(x)) }
```

```
rLCN=function(n,a,c) {
   x=numeric(n)
   for (i in 1:n){
      x[i]=ifelse(runif(1,0,1)<c,rnorm(1,a,1),rnorm(1,0,1)) }
   return(sort(x)) }

rNM=function(n,a,b,c) {
   x=numeric(n)
   for (i in 1:n){
      x[i]=ifelse(runif(1,0,1)<c,rnorm(1,0,1),rnorm(1,a,b)) }
   return(sort(x)) }

dEdge=function(x,a,b){
   return(dnorm(x,0,1)*(1+a*(x^3-3*x)/6+b*(x^4-6*x^2+3)/24)) }
rEdge=function(n,a,b,xl,xu){ #with support (xl,xu)
   wyn=numeric(n)
   e=optimize(function(x)
dEdge(x,a,b),interval=c(xl,xu), maximum=1)$maximum
   d=dEdge(e,a,b)
   for (i in 1:n){
      R1 = runif(1,xlow,xup)
      R2 = runif(1,0,d)
      w = dEdge(R1,a,b)
      while(w<R2){
      R1 = runif(1,xlow,xup)
      R2 = runif(1,0,d)
      w = dEdge(R1,a,b) }
      wyn[i]=R1 }
   return(sort(wyn)) }
```

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Supporting the Age-Period-Cohort model of default rate prediction with interpretable machine learning

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Abstract. Regular short-term forecasting of defaults is a basic activity of a retail portfolio risk manager. From a business perspective, not only the quality of the forecast is significant, but also the understanding of the trends and their driving factors. The vintage analysis and a more advanced Age-Period-Cohort approach are popular tools used for the purpose. The aim of this article is to demonstrate that interpretable machine learning can support the Age-Period-Cohort approach, facilitating forecasting beyond the time range of training data, eliminating the model identification problem and attributing cohort quality to the specific characteristics of loans approved in a given month. The study is based on real consumer finance portfolios from the Polish market.

Keywords: credit risk, macroeconomic impact, age-period-cohort, machine learning, XGBoost, SHAP

JEL: C41, C53, C55, C58, G20, G21

1. Introduction

Default rate prediction is a field of research very important for individual banks, as well as for the stability of the global financial system. This is reflected in the number of international regulations on that matter and the centralisation of loss forecasting units in large international banks. In particular, a part of the risk manager's responsibilities in a retail lending business is short-term forecasting of the default rate and understanding its driving factors.

A typical analysis takes the form of the following process: having received an annual or quarterly loss budget, approved by the corporate management board, the risk manager is obligated to declare whether his/her portfolio is heading above the budget, below it or whether it is on track. If it is off track, he/she must determine if this is due to the portfolio age, the profile of the customers in the portfolio, credit policies, collections policies or the macroeconomic environment. The risk manager must then propose a remediating action (change in the underwriting criteria, promotions in certain sales channels, adjusted pricing, modifications in the collections policies, etc.) to set the forecasted default rate back on track, as determined by the budget.

The data available for the risk manager include credit application data, behavioural data on bank accounts (credit and non-credit behaviour) and data from

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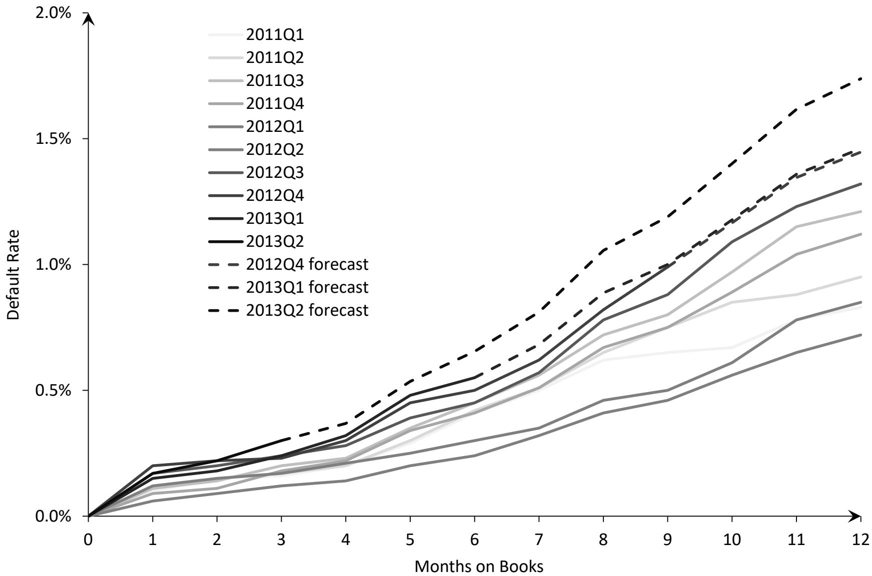
the credit bureau covering information from other financial institutions. Statistical and data management tools include Online Analytical Processing (OLAP), business intelligence reports, statistical classification models (e.g. application scoring used for assessing the creditworthiness of new clients at the moment of credit application, and behaviour scoring used for the assessment of the creditworthiness of clients already in the portfolio). The toolkit also contains portfolio forecasting models (e.g. migration or survival models predicting portfolio evolution). Textbooks explaining thoroughly this classical approach are Lawrence and Salomon (2002) and Siddiqi (2017).

In the recent years, machine learning models have been tested for purposes related to credit risk management (Bracke et al., 2019; Kaszyński et al., 2020). Publications on the success or failure of machine learning used in a real business environment are scarce, and this paper is intended to fill this gap. The study tests the hypothesis that OLAP-based vintage analysis and portfolio forecasting tools based on OLAP can be replaced with interpretable machine learning.

Let us then look in more detail at the practical aspects of default rate prediction. Of all factors affecting the default rate, the effect of portfolio aging is the most treacherous. Defaults take some time to develop, as the most common default trigger is 90 days payment arrears. In the case of new, dynamically growing portfolios, this will cause the numerator of the default rate (number of defaults) to remain low, while the denominator (number of open accounts) will be growing high. This makes unexperienced risk managers think that the credit losses will be below the budgeted level and encourages them to relax credit policies. A few months later it inevitably leads to exploding default rates, with consequences going as far as business closure.

In order to avoid such mistakes, a vintage analysis was developed (Siarka, 2011), together with business intelligence solutions supporting it. The main idea of vintage analysis is to analyse default rates by cohort (the month of booking). This way, credit risk managers can clearly see the default rates grow with the cohort age. Furthermore, they can compare relative risks of different cohorts, relating them to sales campaigns, characteristics of incoming clients or underwriting policies applied at that time, which is illustrated by in Figure 1.

Figure 1. Typical chart used for vintage analysis obtained by means of an OLAP cube (pivot table).

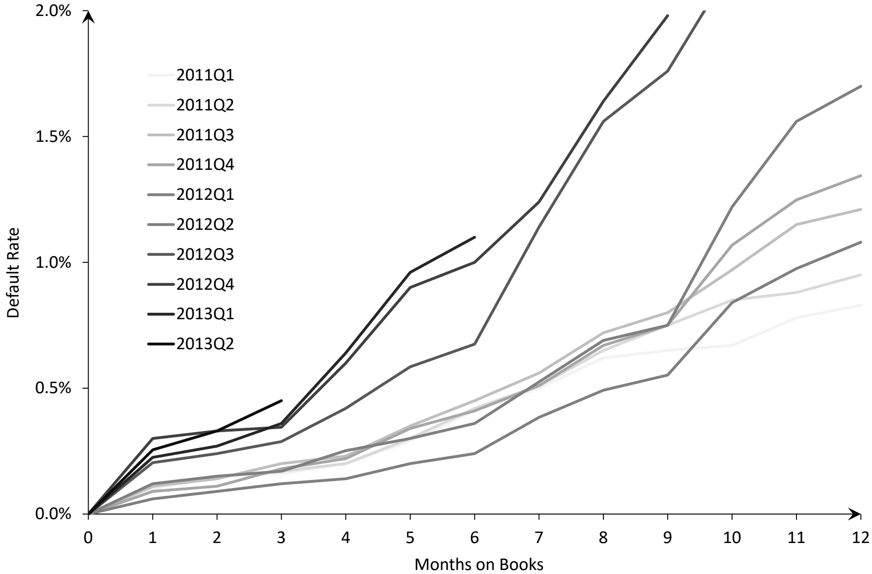


Note. The lines correspond to cohorts (vintages). This can be further segmented based on information available at the time of underwriting using a standard OLAP functionality.
Source: author's work.

Vintage analysis can also support the short-term forecasting of default rates. When the effect of portfolio aging on default rates and the relative differences in risk between cohorts is known, default rates of younger cohorts can be forecasted from the performance of older cohorts. Additional simulations may be prepared assuming changes in future underwriting criteria which provide their estimated impact on future default rates. A simulation run before any changes are implemented prevents serious problems in the future.

External factors like the macroeconomic environment further complicate the picture. A strong and sudden economic crisis can compromise the vintage analysis so that all cohorts are affected at once, each of them being at a different age. This undermines the assumption of roughly proportional default rates for various cohorts, which is a challenge for most vintage-based default rate forecasting tools built with business intelligence solutions as shown in Figure 2.

Figure 2. Vintage analysis distorted by an external macroeconomic shock



Source: author's work.

Macroeconomic factors cannot be ignored even in a non-crisis environment. Recently implemented accounting rules on credit risk provisions (IFRS 9, introduced in 2018) require credit institutions to forecast credit losses under various macroeconomic scenarios, and default forecasting tools must provide such functionality. For this purpose, a more advanced statistical approach called Age-Period-Cohort (APC) is often applied. In the literature, APC is also called Dual Time Dynamics (Breeden, 2007, 2010; Breeden et al., 2008) or Exogenous-Maturity-Vintage (Borges & Machado, 2022; Forster & Sudjianto, 2013). The link of APC to the vintage analysis is that on top of age and cohort (vintage), it includes an additional dimension of a 'period' which can be linked to the macroeconomic environment.

Extensive research results on APC were published by Breeden (2007, 2010) and Breeden et al. (2008), who also popularised this method and applied it commercially. A typical business application can also be found in Borgues and Machado (2022). It includes a non-parametric estimation of age, period and cohort effects. Then, the estimated period effects are regressed on macroeconomic data and the cohort effects are regressed on parameters of underwriting. The purpose of running these additional regressions is to identify the driving factors of default rates and to provide inputs for their short-term forecasts. This is because an APC model itself is not able to forecast beyond the period on which it was trained.

In order to improve the quality of short-term default rate predictions, some authors investigated the use of advanced techniques of regressing macroeconomic effects obtained from an APC model on officially published macroeconomic indicators (e.g. Gamba-Santamaria et al., 2021 used a vector autoregressive model for that purpose). Other authors embedded simple behavioural data in the APC framework. For example, Babikov (2013) developed a method of integrating a popular behavioural model of loss forecasting based on a migration matrix of delinquency buckets with an APC framework. Finally, researchers explored non-linear versions of an APC model (Strydom, 2017).

Nevertheless, all the aforementioned authors used aggregated rather than account level data to develop their models. The reason is that it is costly and time-consuming to estimate an APC model using classical statistical methods when detailed credit application data are used. Such a model does not meet its main business purpose of supporting monthly portfolio quality reviews and providing short-term forecasts for the daily management of a lending business.

Furthermore, most models published so far fail to identify the root causes of delinquencies and attribute them to specific variables like customer characteristics. This task is left to an analyst who segments vintage analysis or APC models using business intelligence solutions in order to find variables corresponding to various risk profiles. Conclusions and business recommendations depend on the strength of the discovered relationships to the same extent as they do on the presentation skills of individual analysts.

This article demonstrates how the XGBoost machine learning algorithm (Chen & Guestrin, 2016) together with SHAP model explanations (Lundberg & Lee, 2017) can be used to make a decomposition of the observed default rates into age, period and cohort effects, then to identify the underlying macroeconomic and idiosyncratic (customer-related) features and finally, to provide short-term forecasts of the default rate. SHAP model explanations replace the expert judgement of the impact of specific customer characteristics on the default rate. The model can be estimated within a day in a fully automated way, eliminating the issue of long delivery time. The combination of gradient boosting and SHAP was also explained in more detail in Bracke et al. (2019) and Kaszyński et al. (2020).

The article further consists of Section 2, which presents the modelling methodology of an APC model and a new machine-learning model, Section 3, which describes the data used for the research, Section 4, presenting the model evaluation criteria, results and conclusions, and Section 5, which summarises the modelling methodology and demonstrates the stages of the analysis that might be used in any lending business. The latter is the paper's contribution to the development of the field of credit loss forecasting and credit risk management.

2. Model specification

This section describes the traditional APC model and discusses its advantages and disadvantages. Then, the proposed machine learning model is presented and its functionality is compared with APC. Finally, the technical details of the model estimation are provided.

2.1. Age-Period-Cohort model

An APC model is applied to explain various measures (in the OLAP sense) defined on a population, which may be segmented with respect to the origination date and age as the key dimensions. The model is non-parametric and it does not provide forecasts beyond the time range on which it was trained. Results from an APC model are used as inputs for further analysis, which may produce short-term forecasts of the measure in question.

In a credit risk context, APC decomposes an observed default or delinquency rate into effects of the date of the loan origination (also called vintage), portfolio aging (also called months on books – MOB), and the calendar date on which the default rate was reported. The effects of vintage provide information about the quality of the underwriting, which, in turn, depends on the riskiness of the sales channels and the credit policy criteria. The effect of aging results from the contractual maturities of the granted loans, defaults, prepayments and the level of adverse selection due to poor portfolio management. The effect of calendar date is primarily linked to the macroeconomic environment, but it is also impacted by early debt collection policies and regulations, such as payment holidays. Therefore, as already mentioned, further analysis is usually done with business intelligence tools or with statistical means to explain the results obtained from an APC model and to attribute the observed trends in delinquencies/default rates to their root causes.

The general formula of an APC model reads:

$$f\big(m(a,p,c)\big) = \alpha_a + \pi_p + \zeta_c + \varepsilon_{a,p,c}. \tag{1}$$

In this formula, f is a link function – usually a logit, probit or natural logarithm, m is the modelled measure (e.g. the default rate), α_a is a series of coefficients corresponding to the values of age (MOB) a, π_p is a series of coefficients corresponding to reporting dates (periods) p, ζ_c is a series of coefficients corresponding to dates of loan origination (cohorts, vintages) c, $\varepsilon_{a,p,c}$ are error terms with expected values of 0.

In general, no further assumptions are made regarding the distributions of error terms; nevertheless, particular methods used for APC estimation may still use their specific assumptions.

The estimation of an APC model is usually done on aggregated data, i.e. a pivot table producing the measure in question and the number of observations for each combination of a, p, c. Since $a = p - c$, one of the dimensions in this pivot table is redundant. The pivot table must cover consecutive values of period p and cohort c. Then the coefficients of all the values of a, p, c observed in the dataset will be produced by the model. As the model is non-parametric, it is not possible to produce forecasts for the values of a, p, c not present in the development dataset.

The general formula of an APC model poses two identification problems. First, any constant can be added to coefficients α_a and subtracted from π_p or ζ_c without any change in the model fit. This issue is purely technical and it has no impact on the practical interpretation of the results, as coefficients α_a, π_p, ζ_c can be presented in such a way that their mean value is zero. However, the second model identification issue is serious. Note that as $a - p + c = 0$, for any number τ we can obtain an alternative set of coefficients producing the same prediction, but differing by a linear trend from their original versions:

$$\alpha_a + \pi_p + \zeta_c = \alpha_a + \pi_p + \zeta_c + \tau(a - p + c) = (\alpha_a + \tau a) + \left(\pi_p - \tau p\right) + (\zeta_c + \tau c). \tag{2}$$

From the user's perspective, this poses a serious problem. The user of an APC model would want to know if the recent trend in the modelled variable (e.g. default rate) is caused by a trend in cohort quality (e.g. caused by underwriting criteria), portfolio age or a trend in external factors. This has an obvious impact on the action plan that the risk manager would propose. However, due to the model identification issue trends in the model, the coefficients can be freely manipulated by an analyst estimating the model. The data provide no answer as to which version of the coefficients is correct.

It should also be noted that the model identification issue does not depend on link function f or any additional assumptions relating to the distribution of error terms. Therefore, no estimation technique can solve this problem unless additional data are provided or additional assumptions are made (Forster & Sudjianto, 2013).

To sum up, the main advantage of an APC model is its simplicity and the fact that it involves very few upfront assumptions. The disadvantages include the identification problem and inability to provide reliable forecasts for cohorts, ages or periods going beyond the development dataset.

2.2. The idea of a challenger model

A tempting modification of an APC model would be to use an application score instead of the cohort indicator. It assumes that the application score summarises all the relevant information about the credit risk, and the difference of the average credit scores for the given cohorts reflects the differences in the quality of the underwriting. This reasoning, however, is flawed for a number of reasons. Firstly, underwriting is often based on a few scorecards (e.g. separate models for new and existing clients, separate models for clients with or without a credit bureau record) that are rarely consistently calibrated, making their resulting scores incomparable. Secondly, the sales channel is not usually included in the application scorecard, yet it might be a significant risk factor. Thirdly, the application scorecards may be frequently modified, thus making some cohorts incomparable by considering these scores alone.

In light of the arguments above, it is tempting to take all the relevant data captured at the time of application (sales channel, socio-demographics, credit bureau variables) and estimate an equivalent of an APC model with such raw data. These data are usually easily available, as they are produced for a periodical review of the application scorecards and for business intelligence reporting. Nevertheless, developing such a model with classical means, even without a strict validation process, can take several weeks, if not months. The APC model, on the other hand, is supposed to provide quick answers within days. Once set up, it takes only a few hours to estimate such a model and produce a summary report.

Interpretable machine learning can help improve the delivery time of the analysis above. The idea is to consider the measure in question (in this case the default rate) at the level of individual observation, so that it becomes a zero-one variable. Then, interpretable machine learning is run with a logit link function on the application data, the account age (MOB), and the indicator of the period, or, in another variant, on a pre-defined set of macroeconomic variables. The SHAP algorithm can then attribute the prediction to the period, age, and application data. As the SHAP algorithm provides additive attributions, the SHAP values for the application data can be added up for each observation to produce an equivalent of an application score. Then, the average of this application score equivalent over a cohort (vintage) can be taken to represent the quality of the underwriting in a given cohort. Similarly, the sum of the SHAP values for all the macroeconomic variables for a given observation provides a total attribution of the modelled measure to the external environment. A vector of averages of these SHAP values by period provides an equivalent of the period coefficients in an APC model.

The use of a detailed application and macroeconomic data makes it possible to produce forecasts beyond the development dataset. Reasonable assumptions about the cohort quality can be made. They can be based on e.g. the sales budget by channel, trends in underlying customer characteristics such as past delinquencies, debt to income etc., and based on the expected changes in the credit policy. Similarly, macroeconomic scenarios can be used to make forecasts of the period coefficients. Finally, age coefficients can simply be extrapolated, as they flatten out with age (as demonstrated in Figure 7).

Finally, a detailed attribution of the measure in question to a particular application or macroeconomic data indicates which parameters of the incoming applicants should be monitored with classical business intelligence tools and which macroeconomic variables should be forecasted in macroeconomic scenarios.

Taking the above into consideration, the challenger model proposed here should be able to eliminate both of the indicated drawbacks of a simple APC model, to provide additional insight into the root cause of the identified trends of default or delinquency rates and to deliver a meaningful final report within a few of hours, once it is set up.

2.3. Specification of the challenger model

In this section, the results of the following algorithm of the proposed model are presented: an XGBoost model is run with logit output (option 'binary:logitraw') on a training sample. The modelled outcome is 1 for the accounts defaulting in the next calendar month, and 0 otherwise. The explanatory variables are: idiosyncratic predictors gathered on application date $X(a)$ for account a, macroeconomic variables $M(t)$ for observation date t, and months on books $mob(a,t)$. The model produces $\widehat{logit}_D\big(X(a), M(t), mob(a,t)\big)$, which is then converted to the probability of a default occurring in the following month by the formula below:

$$PD\big(X(a), mob(a,t), M(t)\big) = \frac{\exp\Big(\widehat{logit}_D\big(X(a), M(t), mob(a,t)\big)\Big)}{1+\exp\Big(\widehat{logit}_D\big(X(a), M(t), mob(a,t)\big)\Big)}. \quad (3)$$

The model is run in the variants presented in Table 1.

Table 1. The applied model variants

Lagged macroeconomic variables (AL)	$M(t)$ consists of macroeconomic data with 6 lags
Coincident macroeconomic variables (AC)	$M(t)$ consists of macroeconomic data without lags
Dummy variables (AD)	$M(t)$ consists of dummy variables for the calendar month
No macroeconomic variables (AN)	

Source: author's work.

The model corresponds to an APC framework in a sense that the MOB has the meaning of age, the macroeconomic variables describe the impact of the 'period', and the idiosyncratic information gathered at the time of credit application corresponds to the quality of the cohort.

The replacement of cohort indictors with idiosyncratic application data eliminates the identification problem of an APC-based approach. It is subject to assumption, though, that all the relevant cohort quality parameters are captured by these idiosyncratic data.

2.4. Grid search

The learning parameters have been optimised separately for each model variant, and only the results of these optimum models are presented in this paper. In order to optimise the learning parameters, the following algorithm was run: depth of trees – values 2, 3 and 4 were tested, within each depth, learning rates 1.0, 0.5, 0.25 were tested, within each learning rate, the number of trees of 40, 80, 160 were tested.

If the Gini index on the test sample was improved by at least 0.01 from the recently memorised best set of parameters, the old set of learning parameters was discarded, and the new one was remembered.

There is no random (bagging) element allowed in the model estimation, as financial institutions and their regulators prefer to have no random components in their models.

2.5. Explanation of the predictions

The TreeSHAP algorithm implemented in the Python SHAP package was applied to explain the aforementioned XGBoost model. It provided for the training, testing and out-of-time samples:

- an additive explanation of the predictions (logit of default) for individual observations and for the entire sample;
- a summary of the feature (predictor) importance;
- the relationship between the predictors and their SHAP values.

The above is in line with the practice already established in the financial industry (Bracke et al., 2019; Kaszyński et al., 2020). More on the SHAP algorithm can be found in Lundberg and Lee (2017).

Note that the SHAP values can be calculated for data out of the training sample. Therefore, once the model is developed, its SHAP values may be applied to many monthly snapshots of fresh data without the need to re-estimate the formula. This functionality is demonstrated in Section 4.

2.6. Model constraints

In order to improve interpretability, the XGBoost models were run with interaction constraints on all $X(a)$, $mob(a,t)$ and $M(t)$ variables. None of these variables were allowed to interact with each other. Similarly, following a common business practice in scorecard development, monotonicity constraints were applied to the $X(a)$ and $M(t)$ variables, except for the categorical ones. Monotonicity constraints mean that the probability of default in the model can only increase in the direction indicated by a subject matter expert. Constraints imposed on macroeconomic variables are presented in Table 2. All lagged variables share an indicated direction of their base variable.

Table 2. Monotonicity constraints imposed on macroeconomic variables

Variable	Description	Sign
Bankruptcies	New consumer bankruptcies in a given month	+
Deaths	New deaths reported in a given month	+
UnemployedStock	Number of registered unemployed, end of a given month	+
UnemployedRate	Registered unemployment rate	+
UnemployedNew	Newly registered unemployed in a given month	+
UnemployedNewRepeat	Newly registered unemployed in a given month who were unemployed before	+
JobOffersNew	New job offers registered in a given month	–
JobOffersNewPrivate	New job offers registered in a given month, private sector	–
JobOffersEOM	Open job offers on month-end	–
MeanSalaryEnt	Mean salary in the enterprise sector	–
CPI	Consumer price index, change year on year	+
CCI_curent	Consumer Confidence Index, current status	–
CCI_leading	Consumer Confidence Index, future outlook	–
CCI_finance	Consumer Confidence Index, household finances	–
CCI_country	Consumer Confidence Index, economic situation of a country	–
CCI_cpi	Consumer Confidence Index, inflation outlook	–
CCI_unemployment	Consumer Confidence Index, unemployment outlook (inverted sign)	–
CCI_purchases	Consumer Confidence Index, propensity for major purchases	–
CCI_savings	Consumer Confidence Index, savings propensity	–

Source: author's work.

It should be noted, however, that these additional regularisation constraints are feasible without much compromise on the part of the predictive power, because the input data were already carefully prepared, i.e. most of the interactions between the raw variables were captured in the process of constructing predictors $X(a)$.

2.7. Implied macroeconomic factors (period coefficients)

Implied macroeconomic factors, called coefficients of periods in the classical APC approach, can be inferred from SHAP values. Having dummy variables for each calendar month t as the only set of external variables $M(t)$, we can calculate their impact on the logit of the default in the development sample. The impact is measured by the SHAP value of the respective dummy variables. The mean value of the SHAPs for observations with a dummy equal to 1 was calculated. Then, the mean value of the SHAPs for observations with dummy equal to 0 was subtracted from the result. In this way, the implied macroeconomic factor was obtained for each observation month in the training sample.

In this article, the implied macroeconomic factors were compared with the weight of evidence of the calendar month in the training sample. The weight of evidence (WoE) corresponds to the coefficients of univariate logistic regression of the modelled default on the categorical calendar month plus a normalisation constant, making it independent from the choice of the reference category. The weight of the evidence for calendar month t is defined as (Siddiqi, 2017)

$$WoE(t) = \log\bigl(pdf_d(t)/pdf_n(t)\bigr), \tag{4}$$

where pdf_d and pdf_n are probability distribution functions of the defaults and non-defaults, respectively for the analysed portfolio and sample.

Both the implied macroeconomic factor and weight of evidence are presented on the same logit scale. This comparison visually demonstrates to what extent the variance of the default rates is explained by the calendar month, and to what extent other predictors in the model are playing their role. Such a comparison of the score value assigned to a certain category to its WoE is a standard assessment procedure of credit scorecards (Siddiqi, 2017).

2.8. The quality of underwriting (cohort coefficients)

The SHAP values for individual predictors add up to the total predicted logit of default. Separating the SHAP values for static (application) features and adding them up provides a close equivalent of a traditional application score (expressed in a logit scale). Furthermore, averaging this score for the whole cohort provides a measure of the underwriting quality, which is called a cohort coefficient in the APC approach.

As accounts close, either due to prepayment or due to contractual maturity, the distribution of the application data for a given cohort changes along with the months on books. Therefore, the impact of a specific cohort (vintage) on the portfolio quality may depend on the MOB. The quality of the underwriting presented in this article should be understood in the context of a specific portfolio sample.

3. Data

This section describes the data obtained for the research and the sample selection for the development of a machine-learning model.

3.1. Data obtained for research

The gathered data correspond to a typical dataset available in a lending institution for credit risk analysis. It consists of 40 monthly portfolio snapshots between (and including) two dates: T_S and T_E. The records contain an opening date, months on books and a date of default for the defaulted accounts. In these data, accounts never cure from default. The data also contain the application records: the socio-demographics and the summary of the credit bureau reports (e.g. the number of delinquent loans or the number of credit inquiries), altogether 27 potential idiosyncratic predictors. The data are fully anonymised.

Additionally, for the same period between (and including) T_S and T_E, selected macroeconomic data were obtained from 'Statistical Bulletins' (Pol. 'Biuletyny statystyczne'), available on the Statistics Poland portal,[1] including lagged data up to 6 months.

The data cover four different portfolios with different characteristics in terms of maturity, prepayment and default risk. Furthermore, the important idiosyncratic application data differ considerably in their distribution. Therefore, repeating the modelling procedure on these four portfolios guarantees that the modelling results were not obtained accidentally, and that one can draw general conclusions from the performance of the proposed methodology.

3.2. Sample construction

For each portfolio, the following samples were built:

- A training and testing sample (50%/50%) of the portfolio on the development window. An equal size of a training and testing sample was used to make relative forecast errors comparable. Using a different proportion results in a higher forecast error on a smaller sample due to the higher variance of the observed

[1] See: https://stat.gov.pl/obszary-tematyczne/inne-opracowania/informacje-o-sytuacji-spoleczno-gospodarczej/biuletyn-statystyczny-nr-72023,4,140.html.

default rates for any calendar month, which is unrelated to the quality of the model and its explanatory variables. The large number of observations in the available dataset allowed this equal split rather than a 70%/30% one, commonly used for smaller portfolios;

- An out-of-time sample (OOT). Its purpose is to test how accurately the proposed model can forecast beyond the time range of the development sample. This is in line with a common business practice of backtesting loss forecast models.

The algorithm procedure of sample selection involves:

- Preparing a Cartesian product of all dates between (and including) T_S and T_E with a set of account ids ever open between these dates. Each observation is a pair of an account id and an observation date;
- Dropping from this Cartesian product the observations where the account was closed or defaulted on or before the observation date. Observations with accounts not yet open on the observation date should also be dropped.

The two steps above are consistent with taking a representative sample of an open portfolio for all observation dates between (and including) T_S and T_E, which, again, is a common business practice in credit risk modelling. The subsequent steps are:

- Selecting an interim censoring date T_I six months before end date T_E. No data after the interim date are available for the model development. It applies to the predictors, outcome and macroeconomic data;
- Forming the out-of-time sample from all the observations with an observation date on or after T_I;

The first two steps above involve blindfolding the model to all the information coming on or after T_I. An out-of-time sample will be used to backtest the model, i.e. to check if it is able to forecast default rates over the period between T_I and T_E, for which no prior information was received.

- Forming the development sample from 50% of the observations from the remaining set (observation date before T_I), forming the test sample from the rest.

The predictors were taken as of the observation date. They include static (application) data, account age (months on books) and lagged macroeconomic variables. The target variable (default or not) was taken as of the calendar month following the observation date.

In the next step, all observations in the development sample with non-default outcome were down-sampled in order to reduce the computational burden. All observations with a default status were left in the development sample. When calculating predictions from the model, a constant is added to the predicted logit of default to calibrate the default rate forecast to the population before down-sampling.

Table 3 summarises the number of defaults in each sample, which is critical for the performance of any form of logistic regression. The total number of observations is not shown, so that confidential corporate information is not disclosed.

Table 3. Sample counts (number of defaults)

Sample	P1	P2	P3	P4
Training	6015	7784	5078	8505
Test	6040	8027	5177	8679
OOT	4406	5479	4065	7416

Note. P – portfolio.
Source: author's work.

4. Results

This section is devoted to the presentation of the model evaluation measures and model evaluation results, followed by conclusions on the degree to which the proposed model meets the expectations. On the technical side, in all of the estimated variants, the grid search algorithm chose depth 2, learning rate 1.0 and 40 trees, and only the results for models obtained with these parameters are presented.

4.1. Model evaluation measures

The model evaluation measures presented in this section are appropriate for the proposed machine learning methods and not relevant to the standard APC approach. They describe how accurately the model is able to predict default rates beyond the period on which it was developed, and how exhaustively default rates can be explained with the underlying detailed idiosyncratic and macroeconomic data. None of these is a functionality of the standard APC approach, therefore classical APC is not included in the comparison.

For each calendar month, the portfolio (P1–P4) and the sample (training, test, OOT), the following measures were calculated and compared:

- forecasted default rate $\widehat{DR}(t)$ based on model predictions, defined as an average of $PD\big(X(a), mob(a,t), M(t)\big)$ for all accounts a in the sample, which were open in calendar month t;
- realised default rate $DR(t)$, defined as the ratio of:
  - the number of accounts in the sample that were open in calendar month t in the denominator,
  - the number of such accounts that defaulted in the next calendar month in the numerator.

The quality of fit is evaluated with a relative forecast error, given by a simple formula easily understood by business users of the proposed models:

$$RelativeError = \frac{\sum_t |DR(t) - \widehat{DR}(t)|}{\sum_t DR(t)}. \tag{5}$$

As the default rate forecast does not have the same mean value over time t as the default rate realisation, it is impractical to use R^2 as a measure of the model fit. It may yield values higher than 1 or lower than 0 – and in fact it often does. As the purpose of this article is to compare various approaches, it is important that the quality of fit has the same denominator for all of them. This is why the realisation of the default rate is used in the denominator rather than in its forecast.

Even though the quality of the default rate forecast is primarily sought, the quality of the default/non-default separation was also measured with a Gini index, which is a standard approach in the consumer-lending industry.

4.2. Summary of the results

Tables 4 and 5 present the relative forecast errors and the Gini indices, respectively.

Table 4. Relative forecast errors

Portfolio/approach	Training	Test	OOT
P1/AL	5.8%	8.5%	14.0%
P1/AC	6.6%	8.6%	17.0%
P1/AD	8.5%	11.0%	14.8%
P1/AN	12.2%	11.5%	12.7%
P2/AL	6.5%	6.3%	5.4%
P2/AC	6.7%	8.0%	7.3%
P2/AD	9.0%	9.8%	9.6%
P2/AN	12.1%	11.6%	8.3%
P3/AL	7.5%	8.8%	7.5%
P3/AC	6.8%	9.7%	14.7%
P3/AD	10.5%	11.9%	9.0%
P3/AN	13.7%	14.5%	8.8%
P4/AL	6.1%	6.4%	2.5%
P4/AC	6.1%	7.4%	2.7%
P4/AD	6.8%	8.2%	16.2%
P4/AN	11.0%	11.2%	15.1%

Source: author's work.

Table 5. Gini indices

Portfolio/variant	Training	Test	OOT
P1/AL	62%	61%	51%
P1/AC	62%	61%	51%
P1/AD	62%	61%	51%
P1/AN	62%	61%	51%
P2/AL	66%	65%	59%
P2/AC	66%	65%	58%
P2/AD	66%	65%	59%
P2/AN	66%	65%	59%
P3/AL	67%	65%	59%
P3/AC	67%	65%	59%
P3/AD	67%	65%	59%
P3/AN	67%	65%	59%
P4/AL	58%	57%	54%
P4/AC	58%	57%	54%
P4/AD	58%	57%	54%
P4/AN	58%	57%	54%

Source: author's work.

The model performance measures on the test and the training sample provide information about the model fit. A model overfit can also be detected if the measures are considerably better on the training sample than on the test sample. On the other hand, the model performance on the OOT sample says if the model is able to extrapolate its forecast beyond the time scope of the training sample. The results show no overfit with respect to idiosyncratic data, while some overfit is observed with respect to macroeconomic data (or period coefficients), reflected in higher relative forecast errors on the test sample compared to the training sample. Furthermore, despite some drop on the out-of-time sample, the Gini indices remain strong. It means that the model is able to detect relationships in the idiosyncratic data which are stable over time.

It is quite surprising to see that the Gini index does not really depend on the approach to macroeconomic data, while the relative forecast error depends on it strongly. Approach AN without any period indicators and without macroeconomic data performs worst of all on the training and test samples. Approach AL with lagged macroeconomic data is able to provide a very accurate forecast, for example for portfolios P2 and P4. However, as shown in Table 6, the proposed algorithm is not very good at selecting macroeconomic variables consistently. This indicates the need to perform a reduction of dimensionality of macroeconomic variables and feature engineering in this area based on expert judgement, e.g. introducing the moving averages or differences of some macroeconomic variables. In this context, it should be noted that even though the number of observations provided to the machine-learning algorithm is large, the effective dimension of the macroeconomic data equals the number of months in the training sample, which is 34. The presented machine-learning algorithm is based on an already pre-selected set of 19 variables, which with 6 lags each makes a total of 133 candidate variables. The right or wrong choice of macroeconomic variables may be the reason behind the inconsistent performance of model variants with macroeconomic data on the OOT sample.

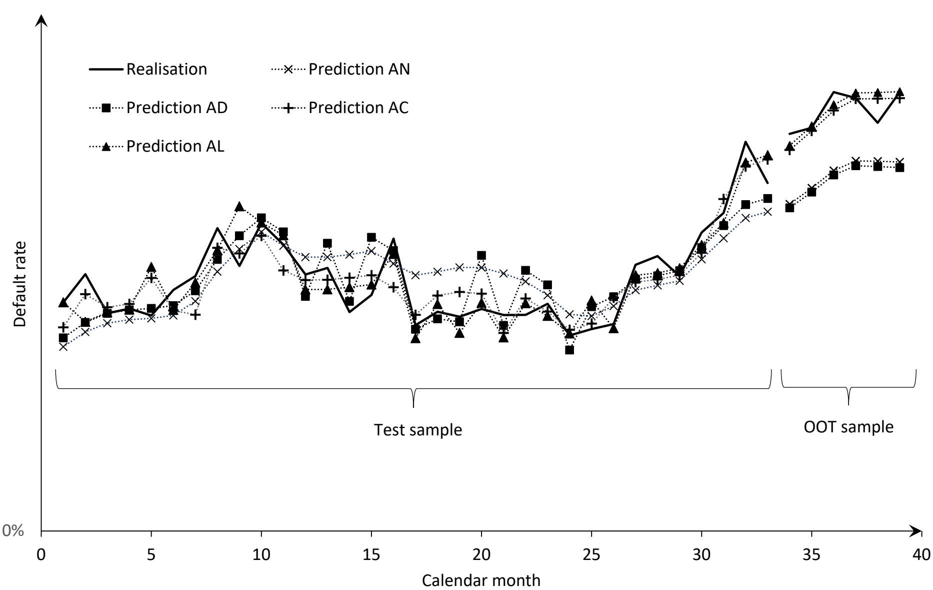
Table 6. Automatically selected macroeconomic variables

Portfolio/variant	Variant with coincident variables
P1/AC	UnemployedNew, UnemployedNewRepeat, JobOffersNewPrivate, CPI, CCI_current, CCI_cpi
P2/AC	UnemployedNewRepeat, MeanSalaryEnt, CCI_savings
P3/AC	UnemployedNewRepeat, JobOffersNew, MeanSalaryEnt, CPI
P4/AC	UnemployedNewRepeat, CPI, CCI_savings
Portfolio/ variant	Variant with lagged variables
P1/AL	Deaths_5, UnemployedNewRepeat_0, UnemployedNewRepeat_5, CPI_1, CPI_3,
P2/AL	UnemployedNewRepeat_3, MeanSalaryEnt_1, CPI_1, CCI_savings_1
P3/AL	UnemployedNewRepeat_2, MeanSalaryEnt_1, CPI_0, CCI_cpi_4
P4/AL	UnemployedNewRepeat_0, CPI_1, CCI_savings_1

Source: author's work.

Figure 3 presents the predictions of the default rate and its realisations. No scale is shown on the Y axis so that the true default rate of the data provider is not disclosed for legal reasons.

Figure 3. Predictions and realisation for portfolio P4, test and OOT samples.
The OOT sample starts to the right of the visible gap in lines, months 34–40

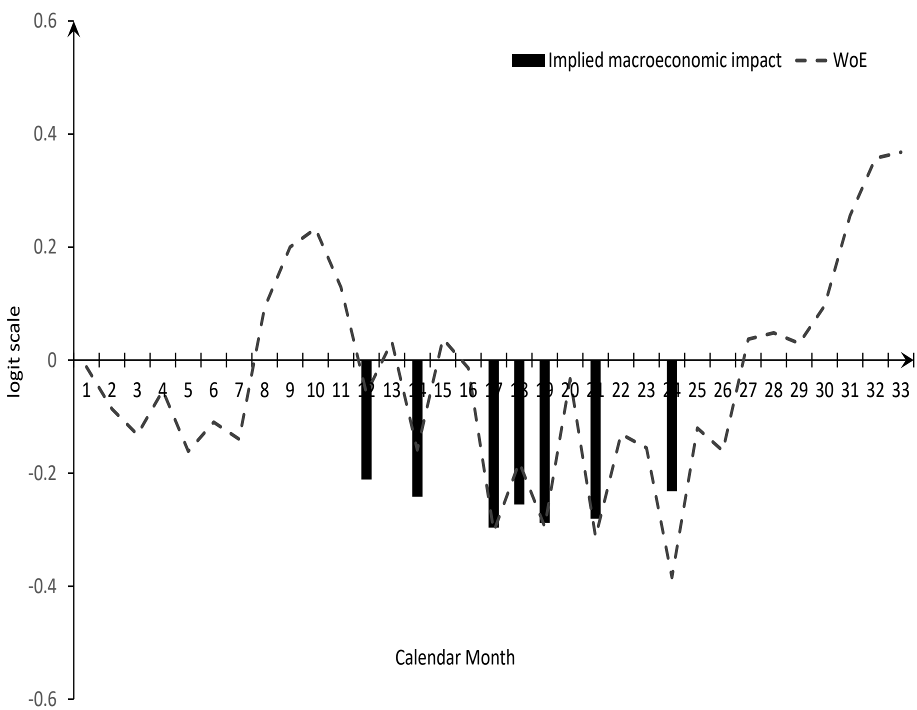


Source: author's work.

As Figure 3 demonstrates, variant AN ignores the improving macroeconomic environment between months 10 and 25 as well as its worsening after month 30. Variant AD clearly overfits the random fluctuations of the training sample (shown in Figure 2), but makes a smaller systematic error on the test sample. Both the AN and AD variants perform poorly on the OOT sample, as variant AD was not provided with any macroeconomic scenario from month 35 onwards. Not surprisingly, it shows a nearly identical forecast as AN on the OOT sample. The variants with true macroeconomic data, AC and AL, perform really well on both test and OOT samples, at least for portfolio P4. This, despite the difficulties mentioned in Section 4.2, confirms the technical possibility to build good machine-learning models with macroeconomic data, as required by IFRS 9 regulations and stress test requirements imposed by supervisors of financial systems.

Figure 4 shows how the model with dummy variables produced implied macroeconomic factors for portfolio P4.

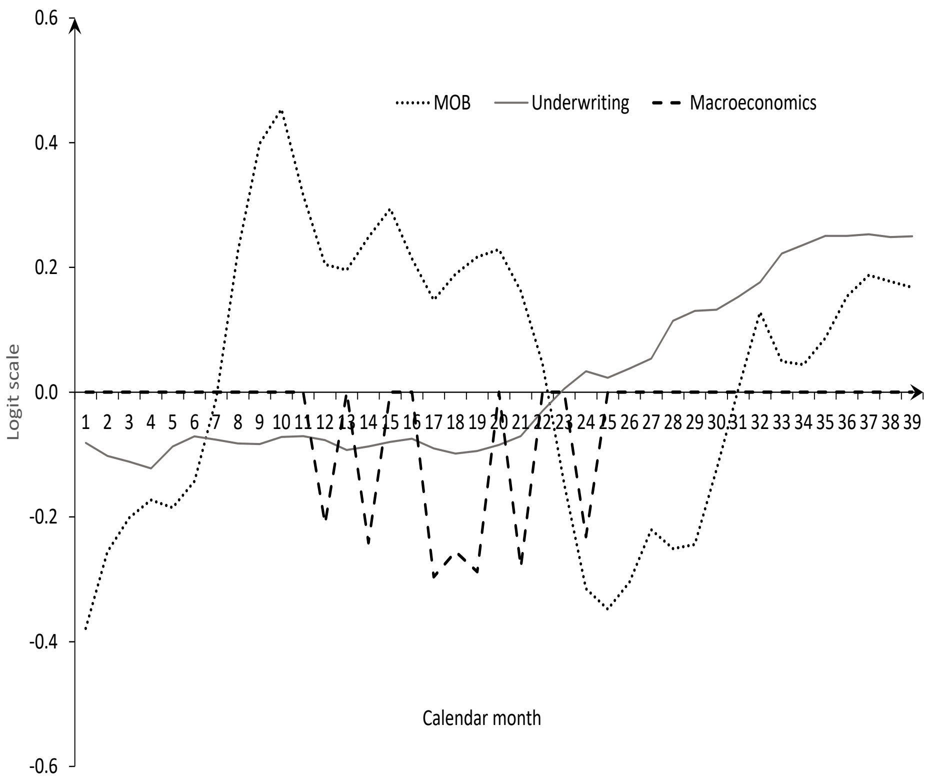
Figure 4. Implied macroeconomic factors by reporting month – portfolio P4, training sample



Source: author's work.

The improvement of the macroeconomic environment in months 12–24 was correctly identified, and furthermore aligned with WoE in this period. The model did not attribute an increased default rate to the macroeconomic situation in months 30 to 33. Instead, it was attributed to the relaxed underwriting policy and portfolio age, as shown in Figure 5.

Figure 5. Decomposition of default rate prediction for each reporting month, portfolio P4, variant AD, test and OOT samples

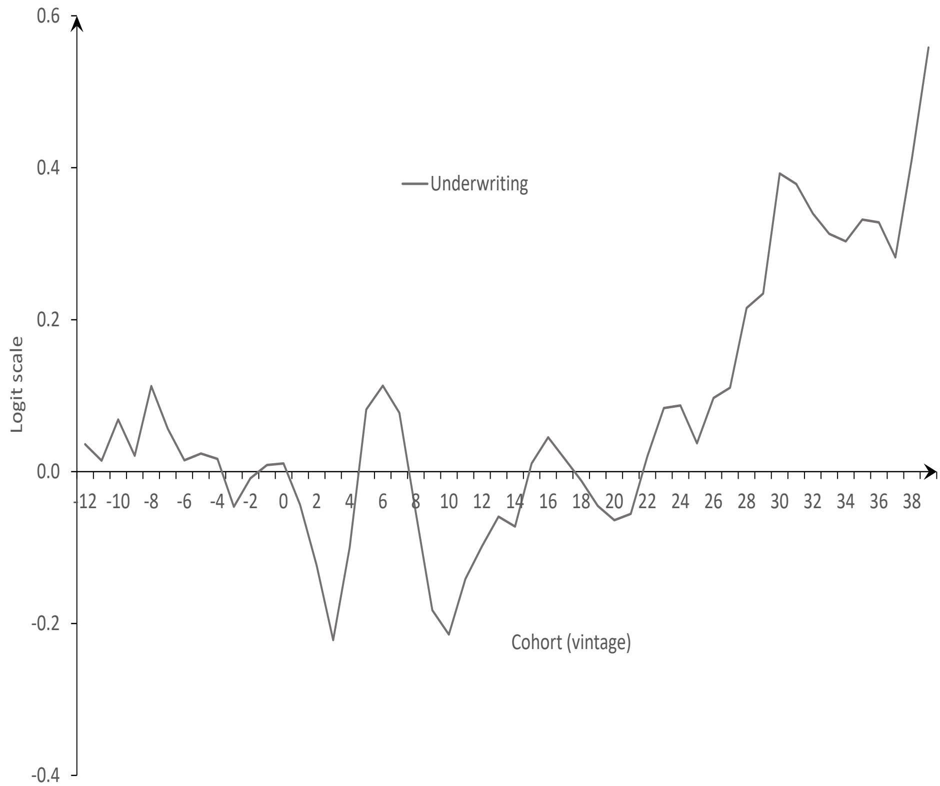


Source: author's work.

This decomposition urges the risk manager to promptly review the underwriting criteria, as the negative impact of bad incoming population was temporarily offset by a relatively young portfolio age in months 24 to 29. This compensating effect ended in months 30 to 35, which resulted in an observed default rate increase in that period.

A better and more traditional way of presenting the quality of underwriting is to plot its dependence on the month of booking (also called a vintage or cohort). An example is shown for portfolio P4 in Figure 6. It was also successfully determined for the OOT sample and for cohorts preceding the observation months (labelled with a negative sign). Note that higher values indicate a higher risk of default due to the relaxation of the credit policy.

Figure 6. Estimated quality of underwriting by cohort, portfolio P4, variant AD, test and OOT samples

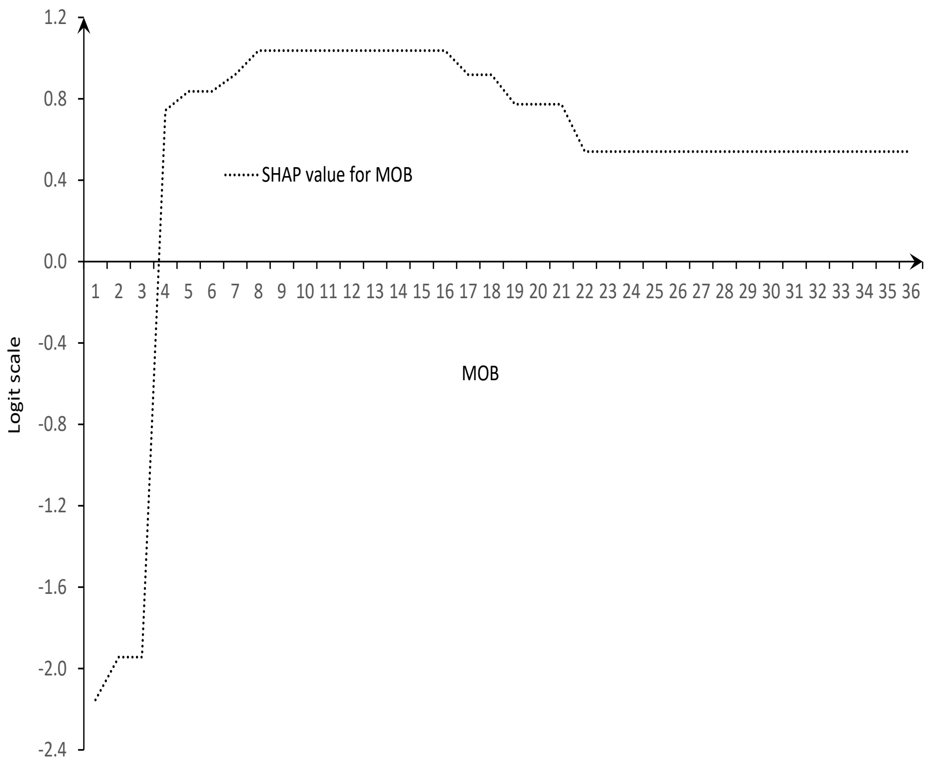


Source: author's work.

Figure 6 shows that the decrease in the credit risk quality of recently booked loans is considerable. Compared to this, Figure 5 does not expose it as much as it mixes the impact of the old and new cohorts for the same reporting month (called period in the APC approach). Here we see an increase of risk by 0.6 on a logit scale between months 22 and 40, which corresponds to the increase of the predicted default rate 1.8 times.

The impact of portfolio aging on the logit of the probability of default is shown in Figure 7 for portfolio P4. The shape of the obtained curve corresponds with that presented in the literature (Breeden, 2007; Borgues and Machado, 2022). Looking at the span of the SHAP values for various MOBs, we can see why MOB is such an important driving factor of default rate prediction, and why it is so dangerous to omit it, as mentioned in Section 1. The span of three logit units accords with the 20-fold difference in the risk of default. This is compared to the span attributed to the cohort of 0.8 (Figure 6), which is in agreement with the default risk increase by a factor of 2. The impact of the macroeconomic environment, much valued in IFRS 9 regulations and stress-testing requirements of the banking supervision worldwide, has the span of only 0.3 (Figure 4), corresponding to the 1.3-fold difference in default risk.

Figure 7. Impact of MOB on the SHAP value, portfolio P4, variant AD, training sample

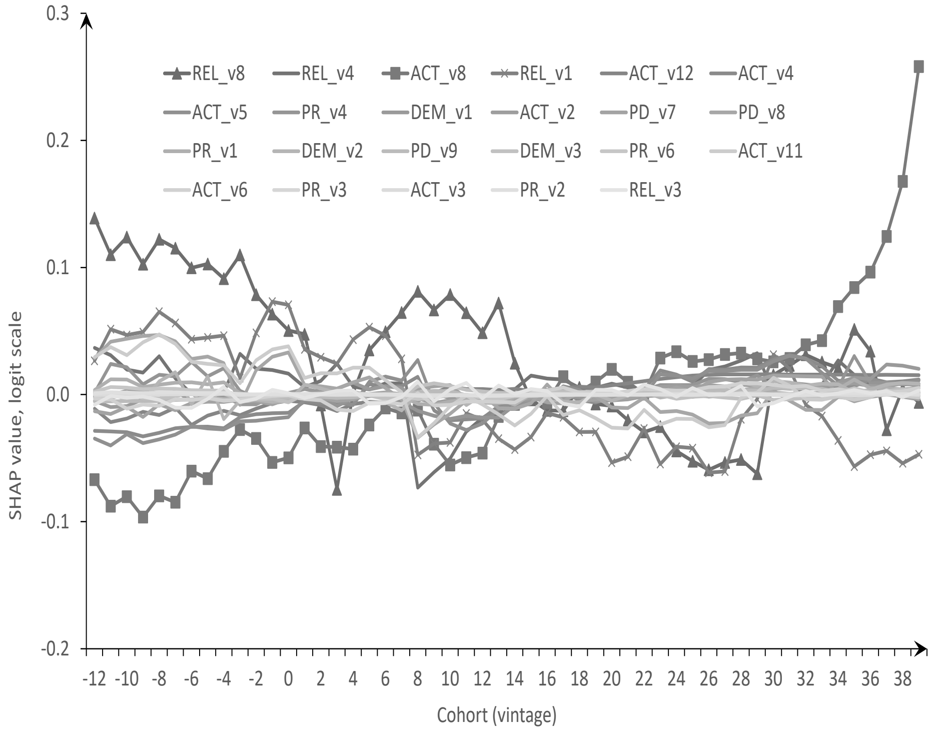


Source: author's work.

Finally, Figure 8 illustrates how the impact of the underwriting quality on the predicted default rate can be decomposed to individual variables, providing a useful insight to a risk manager who can correct the underwriting criteria to meet the business targets. The chart clearly shows that the worsening of the underwriting quality in cohorts 34 to 39 is due to variable ACT_v8, which exhibits a continuous trend of increasing contribution to the default risk. This trend is accelerating from month 30 onwards. Other idiosyncratic variables have a much lower and more temporary impact.

Fortunately for the risk manager in charge of this portfolio, this pattern of a single variable getting out of control can be easily corrected by imposing a single additional underwriting criterion on this variable, which would likely be a recommended action.

Figure 8. Decomposition of the quality of underwriting, portfolio P4, variant AD, test and OOT samples



Source: author's work.

5. Conclusions

Interpretable machine-learning applied in an APC framework can combine short-term portfolio forecasting with a useful insight into the driving factors of the default rate and their trends. It is quick to set up and run, and it requires little intervention from an analyst. It can partially replace traditional monthly portfolio quality reviews based on business intelligence solutions, and indicate which underwriting features should be tracked with more conventional reporting. The proposed procedure reads as follows:

- prepare a datamart consisting of application data, monthly delinquency and default data, and update it monthly;
- prepare a datamart with macroeconomic variables and update it monthly;
- prepare a sample as described in Section 3.2, without the out-of-time part;
- estimate the model as explained in Sections 2.3–2.8, considering version with dummy variables (AD);
- prepare decomposition charts (Figures 2, 3, 4, 5, 6);
- based on Figure 2, attempt to identify the macroeconomic variables showing a similar time pattern;
- re-estimate the model in version AC (or AL) with shortlisted macroeconomic variables;
- prepare decomposition charts again (Figures 2, 3, 4, 5, 6);
- prepare a short-term default rate/delinquency rate forecast with a macroeconomic scenario;
- prepare your write-up, conclusions and recommendations for the management of your company; some guidelines may be found in Breeden (2010);
- store your results and forecasts for out-of-time testing to be performed a few months later.

The two-step estimation (AD and then AC or AL) is recommended, as the methodology tested in this article has limited capacity to identify the macroeconomic variables driving portfolio performance. Automating the process of macroeconomic variables selection by means of imposing certain regularisation criteria (e.g. unit root tests, co-integration, etc.) remains an interesting topic for further research.

A limitation of the proposed method consists in its lack of utilising behavioural data. Therefore, its business potential is limited to portfolios of loans without transactional data, such as cash loans or mortgages. Furthermore, it is limited to institutions without current accounts, from which useful behavioural information can be extracted. Thus, the proposed model is practical mostly for specialised non-banking retail lenders. For other lenders it may still serve as a useful benchmark for models applying behavioural data.

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The passing of our mentor, Professor Maria Cieślak – memories[1]

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On Sunday, 23 April 2023, we received very sad news that Professor Maria Cieślak had passed away.

'Our Professor Cieślak' is gone. She was 'our Professor', because for many of us she was a thesis advisor or reviewer of doctoral dissertations, and she supported us in our further scientific careers. She was our teacher and mentor, and showed us the beauty and secrets of science.

Maria Łucja Cieślak was born on 13 December 1933 in Barchlin (Wielkopolskie Voivodship). She graduated high school in Leszno, where her reliability, meticulous work 'from start to finish', honesty and respect for the truth developed. In 1951, she studied at the University of Economics in Katowice. She then moved to Wrocław, where she enrolled at the Faculty of Industry of the then Higher School of Economics. It was that university (now the Wroclaw University of Economics and Business) that she connected her professional life with and where she pursued her

[1] The text uses the farewell speeches given by Prof. Elżbieta Gołata and Prof. Irena E. Kotowska at the funeral of Prof. Maria Cieślak.

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scientific career. In 1956, she earned her master's degree in economics in the field of statistics. Professor Jan Falewicz, the first head of the Department of Statistics of the Higher School of Trade (Wyższa Szkoła Handlowa) in Wrocław (the precursor of the Wroclaw University of Economics and Business), influenced her scientific interests. Her outstanding scientific achievements earned her a proposal to work at the University, which she accepted, starting her career in the Department of Statistics, later transferring to the Department of Econometrics.

In 1964, Professor Maria Cieślak defended her doctoral thesis entitled 'Statistical issues relating to working standards' and earned a doctorate in economic sciences. In 1974, at the initiative of Professor Zdzisław Hellwig, the Department of Forecasting and Theory of Demography was established at the Institute of Economic Cybernetics of the Faculty of National Economy. The management of the department was entrusted to Maria Cieślak. In the same year, in the course of her long-term cooperation with the Institute of Scientific Policy in Warsaw, her monograph entitled 'Models of needs for qualified staff' was published. This was the main achievement for which she obtained her postdoctoral degree in economic sciences. That work, under the modified title of 'Models of demand for qualified staff' was published by PWN (Polish Scientific Publishers) in 1976. Five years later, Maria Cieślak became a professor at the University of Economics. In 1981, the Department of Forecasting and Theory of Demography was transformed into the Chair of Forecasting and Theory of Demography, and in 1995 into the Chair of Forecasting and Economic Analysis. With breaks, the professor was head of these units until 2003. In 1988, Professor Maria Cieślak received the title of full professor.

In addition to the function of the Head of the Chair of Forecasting and Economic Analysis, Professor Maria Cieślak also held other important positions at the University. In those particularly difficult times, she always worked with a sense of responsibility and service to the University's academic community. In the years 1979–1981, she was the vice-dean of the Faculty of National Economy and in 1981–1982 she was the vice-rector for didactics. Due to her perseverance and courage, respect for fundamental values, her openness to others and the changing world, she became an inspiration for many at the time.

In recognition of her distinguished service to the University, on 26 September 2007, the Senate of the Wroclaw University of Economics and Business awarded Maria Cieślak the title of honorary professor.

Professor Cieślak's scientific and organisational activity was not limited to the University. She was a demographer and statistician, and a valued scientific authority. From 1975, for several terms of office she was a member of the Committee of Demographic Sciences of the Polish Academy of Sciences and in 1978 she started serving on the Committee of Statistics and Econometrics of the Polish Academy of Sciences. After 1985, she was appointed member of the Government Population Council. She was also a member of the International Union for the Scientific Study of Population (from 1978). Professor Cieślak was active in the scientific community until her very last days. In 2022, she actively participated in the work of a team appointed by the Bureau of the Polish Academy of Sciences to evaluate the proposal for a demographic strategy prepared by the Ministry of Family and Social Policy.

Due to her established position in the scientific community and her conscientiousness in fulfilling her duties, she was often invited to participate in the activities of various bodies responsible for publishing scientific journals. At her home university, Professor Maria Cieślak chaired the Senate's publishing committee for 11 years. Throughout her career, she was the editor-in-chief of *Argumenta Oeconomica* and *Wrocławski Biuletyn Gospodarczy* (Polish Economic Society, Wrocław Branch), as well as member of the editorial committee of *Studia Demograficzne* and the editorial board of *Przegląd Statystyczny*, *Studia Demograficzne*, *Badania Operacyjne i Decyzje* and *Przegląd Statystyczny Śląska Dolnego i Opolskiego.*

For her outstanding achievements, Professor Maria Cieślak received many awards and distinctions from the state, the scientific community and other institutions. Professor Maria Cieślak was distinguished with the Knight's Cross and Officer's Cross of the Order of Rebirth of Poland, and the Golden Cross of Merit. She also received the Medal of the Committee on National Education and the Badge of the Distinguished Teacher. In 1993, the President of Statistics Poland awarded Professor Cieślak with the Golden Honorary Badge 'For Distinguished Service to the Statistics of the Republic of Poland'. Moreover, she received the following regional awards: a Golden Badge 'For Distinguished Service to Lower Silesia' and a Silver Badge of the Builder of the Legnica-Głogów Copper District.

The Wroclaw University of Economics and Business awarded her the title of the Crystal Graduate (Alumnus), as an appreciation for the many years of her scientific and educational achievements and an expression of admiration for her commitment to the academic community.

Professor Maria Cieślak's scientific and research interests focused mainly on two areas: forecasting and demography. She is considered the creator of the Scientific School of Forecasting, developed at the Wroclaw University of Economics and Business. The school is known for paying attention to the entire forecasting process, not just the forecasting method. This entails placing emphasis on the formulation of the purpose of the forecasting, the predictive indications, the determination of the assumptions underlying the forecasting methods and the examination of the compatibility of these components of the forecasting procedure with a selection of forecasting methods. Professor Cieślak devoted a lot of attention to the critical analysis of forecasting methods, formulating proposals for new forecasting methods and building forecasts of economic and social processes. The culmination of her work in these areas was the publication of the 'Economic forecasting. Methods and Application' manual, which was co-written with a team of scientists and which received an award from the Minister of Education in 1994. The publication addresses the entire forecasting process in a comprehensive and in-depth way. The holistic approach presented in the manual made it a publication used at many Polish universities, serving as a guide for a variety of authors of other textbooks in this field.

Professor Maria Cieślak was also the initiator of the regularly organised 'Forecasting in company management' scientific conference. During the event held in 2014, which marked the 20th anniversary of the Chair of Economic Forecasting and Analysis, when sharing her reflections on her 'adventure' with forecasting, she mentioned Alvin Toffler's book 'The Third Wave' as an inspiration for her scientific exploration in this area.

Professor Cieślak's interest in forecasting was also reflected in forecasting demographic phenomena, as demonstrated by her research on the concepts and modelling of demographic development. The Professor devoted a lot of her attention to the search for ever better research tools and for the means of expressing changes observed in population processes, especially those concerning ageing and the labour market. Her approach to the forecasting of demographic processes corresponded to the latest global trends. An excellent example of the above is her method of forecasting based on spatial-temporal analogies. The research on demographic processes resulted in the publication of a book entitled 'Demography. Methods of Analysis and Forecasting' (1982). The Professor was the scientific editor and co-author of the publication, which received a minister's award and had several editions.

Professor Maria Cieślak was the author or co-author of over 100 scientific papers, 16 monographs, and 18 textbooks and academic books. Additionally, she prepared many research reports commissioned by various state institutions and companies.

Professor Cieślak assisted in the writing of numerous doctoral and post-doctoral dissertations as well as professor monographs. She was the supervisor of 13 PhD and eight post-doctoral candidates. Among her PhD students, three went on to receive the title of full professor. She supervised numerous master's degree dissertations. Even after having formally ended her professional activity, Professor Cieślak was always ready to provide help and advice on various issues. She took special care of her students.

She was held in high esteem and recognised by Polish demographers, statisticians and econometricians. Many researchers felt deep respect for her, thus any positive reviews or opinions expressed by Professor Cieślak were particularly significant and valued among the academic community and a testimony to the recognition of one's scientific achievements. That is the reason why she was often invited to become member of various bodies and entrusted with the role of a reviewer. Professor Cieślak reviewed 53 doctoral dissertations, participated in 32 post-doctoral proceedings, and examined 16 applications for the title of professor. She knew how to appreciate another researcher's commitment and effort. Her reviews were highly substantive and written in a kind way, as she always appreciated an individual's achievements and was always willing to make further suggestions as to how to enhance their study. The Professor showed great respect for other researchers, which was particularly evident at conferences: she always listened carefully to the speeches and her comments and proposals were constructive and useful for the authors. After finishing her work at the Wroclaw University of Economics and Business in 2003, Professor Cieślak started working at the WSB School of Banking, initially in Poznań and later in Wrocław.

Professor Maria Cieślak was scientifically active throughout her life. Even when she retired, she wrote papers and reviewed various scientific works: studies, doctoral dissertations and post-doctoral monographs. Her last paper was published in a collective study entitled 'Society in the Age of Change - Interdisciplinary Studies' in April 2023.

Professor Cieślak's attitude stemmed from an understanding of the role of a scholar, whose duty is to 'help to understand the world'. However, 'whether we use the results of their investigations and how we use them depends on the quality of the society'. Referring to the concept of social capital, she defined the role of scholars in its creation. In her opinion, scientists have 'special obligations towards the individual, the society and the state. These commitments concern the effective, axiological, and fiduciary aspects of their activities and behaviour. The main goal of scientists (...) is to bring good, not material benefits that exceed ordinary personal needs as well as those resulting from work' (Cieślak 2017, p. 13)[2].

Despite her great commitment to scientific work, Professor Cieślak had time for her two passions: reading books and contact with nature. Olga Tokarczuk was one of her favourite writers. Trips to the woods and taking care of her plants in her garden gave her the greatest joy.

[2] Cieślak, M. (2017). Laudacja dla profesor Ireny Elżbiety Kotowskiej. *Studia Demograficzne*, (2), 13–16. https://econjournals.sgh.waw.pl/SD/article/view/1377/1229.

On the one hand, we feel sadness and regret about the passing of a person we loved, who has been with us for many years, who has given so much to each of us, whose care we experienced, and whom we were always able to count on. On the other hand, we have a feeling of gratitude to Professor Cieślak for everything we received from her. She set an excellent example of a great scientist and a human being not indifferent to what both the country and other people are experiencing. For that, we will remain forever grateful.

The Polish scientific community has lost an outstanding scholar and educator. An enlightened, open, kind and creative person.

She will forever remain in our grateful memory.

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